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D-branes in N=2 Liouville theory and its mirror

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ABSTRACT: We study D-branes in the mirror pair $N = 2$ Liouville/supersymmetric $SL(2,\mathbb{R})/U(1)$ coset superconformal field theories. After revisiting the duality between the two models, we build D0, D1 and D2 branes, on the basis of the boundary state construction for the H_3^+ conformal field theory. We also construct D0-branes in an orbifold that rotates the angular direction of the cigar. We show how the poles of correlators associated to localized states and bulk interactions naturally decouple in the one-point functions of localized and extended branes. We stress the role played in the analysis of D-brane spectra by primaries in $SL(2,\mathbb{R})/U(1)$ which are descendents of the parent theory.

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Contents

1. Introduction

We have learned that it is very useful to study non-perturbative objects in string theory, especially when they are related to an implementation of holography. This study has proved instrumental both for understanding gauge theory physics, and for getting to grips with aspects of quantum gravity. In this paper, we will concentrate on constructing boundary conformal field theories for theories with $N = 2$ supersymmetry and central charge $c > 3$. These boundary conformal field theories are arguably the most important missing ingredient in the construction of D-branes in non-compact curved string backgrounds with bulk supersymmetry.

The models we will study in detail are the $N = 2$ Liouville theory [1, 2], and the $SL(2,\mathbb{R})/U(1)$ super-coset [3] theory. These two theories are known to be dual [4, 5](see also [6, 7]), and are mapped to each other by mirror symmetry.

Constructing D-branes in these superconformal field theories will not only increase our knowledge of D-branes in non-minimal $N = 2$ superconformal field theories (see also [10, 11, 12]), but it will also enable us to study holographic dualities in closer detail. There are two conjectured instances of holography where these D-branes are expected to be relevant. The $N = 2$ Liouville/$SL(2,\mathbb{R})/U(1)$ background is is related to a conjectured calculable version of holography [5] which involves a bulk closed superstring background [8] dual to a non-gravitational non-local Little String Theory [9]. Note for instance that D1-branes stretching between NS5-branes (which can be interpreted as the W-bosons of the Little String Theory) can be constructed using these boundary conformal field theories.

Another natural area where the D-branes built in this model are relevant is the study of matrix models for two-dimensional non-critical superstrings [11] and Type 0 strings in a 2D black hole. For the latter case, it was conjectured in [13], following the ideas of [14], that the decoupled theory of $N \to \infty$ $D0$ branes of ZZ type in $N = 2$ Liouville, leads to a version of the matrix model of [15] with the matrix eigenvalues filling symmetrically both sides of the inverted harmonic oscillator potential. This matrix model would be dual to 2D Type $0A$ string theory in the supersymmetric 2D black hole background.

Our paper is structured as follows. We first review in section 2 the bulk theories, and the duality between $N = 2$ Liouville theory and the supersymmetric coset. In section 3 we go on to discuss how to relate H_3^+ (i.e. Euclidean AdS_3) boundary conformal field theories to the $N = 2$ theories under consideration. In the following sections, we then explicitly construct and study D0-, D1- and D2-branes. We pause in section 5 to explain how to extend our result to an angular orbifold of the cigar. In the appendices, we collect several important remarks. One concerns the fact that the $SL(2,\mathbb{R})/U(1)$ super-coset characters are equal to the characters of the $N = 2, c > 3$ Virasoro algebra. Another treats the Fourier transformation of the one-point functions, while a third appendix analyzes the $SL(2,\mathbb{R})$ symmetry of general $N = 2$ $c > 3$ conformal field theories.

2. The bulk

In this section we will study aspects of the bulk theory in which the D-branes studied in this

paper will be embedded. We give original points of view on some of the topics treated in the literature. We first discuss the spectrum of the supersymmetric $SL(2,\mathbb{R})/U(1)$ conformal field theory for concreteness. We then study the nature of the duality between the susy coset and $N=2$ Liouville (as well as its bosonic counterpart). We next review the pole structure of the bulk correlators and comment upon the pole structure of the one-point functions. We finish by suggesting that from the perspective of recent developments in non-rational conformal field theories, the duality of the two theories is a case of dynamics being completely determined by chiral symmetries.

2.1 The bulk spectrum of the supersymmetric coset

We consider a bulk theory with $N=2$ supersymmetry, namely the axial $SL(2,\mathbb{R})/U(1)$ super-coset conformal field theory [3]. We review the algebraic details of this $N=2$ algebra in Appendix A. The central charge of the conformal field theory is

$$c = 3 + \frac{6}{k}\,, \tag{2.1}$$

where k is the level of the parent (total) $SL(2,\mathbb{R})$ current algebra. The primaries $\Phi^j_{m,\bar{m}}$ in the bulk, coming from $SL(2,\mathbb{R})$ primaries with spin j, can belong either to the NS or R sector of the $N=2$ algebra. They have conformal dimensions and $N=2$ R-charges [13, 8]

$$\begin{aligned} \Delta_{j,w,n} &= -\frac{j(j-1)}{k} + \frac{m^2}{k} + \frac{(\delta^R_\pm)^2}{8} \qquad & Q_{j,w,n} &= \frac{2m}{k} + \frac{\delta^R_\pm}{2}\,, \\ \bar{\Delta}_{j,w,n} &= -\frac{j(j-1)}{k} + \frac{\bar{m}^2}{k} + \frac{(\delta^R_\pm)^2}{8} \qquad & \bar{Q}_{j,w,n} &= -\frac{2\bar{m}}{k} - \frac{\delta^R_\pm}{2}\,, \end{aligned} \tag{2.2}$$

where

$$m = \frac{n+kw}{2} \qquad\qquad \bar{m} = -\frac{n-kw}{2}\,, \tag{2.3}$$

with $n, w \in \mathbb{Z}$. The constant $\delta^R_\pm$ is zero in the NS sector and ± 1 in the R sector. The Ramond primaries appear always in pairs due to the double degeneracy of the Ramond vacuum. The numbers $m, \bar{m}$ are the eigenvalues of the left and right elliptic generators of the $SL(2,\mathbb{R})$ Lie algebra. Notice that the conformal dimension $\Delta_{j,w,n}$ is invariant under $j \to -j+1$. The two corresponding primaries are related through

$$\begin{aligned} \Phi^{-j+1}_{m,\bar{m}} &= R^{NS/R^\pm}(-j+1, m, \bar{m})\ \Phi^j_{m,\bar{m}}\,, \\ &= \frac{1}{R^{NS/R^\pm}(j,m,\bar{m})}\ \Phi^j_{m,\bar{m}}\,, \end{aligned} \tag{2.4}$$

where the reflection coefficients $R^{NS/R^\pm}(j,m,\bar{m})$ are given by [13]

$$R^{NS}(j,m,\bar{m}) = \nu^{1-2j}\frac{\Gamma(-2j+1)\Gamma(1+\frac{1-2j}{k})}{\Gamma(2j-1)\Gamma(1-\frac{1-2j}{k})}\frac{\Gamma(j+m)\Gamma(j-\bar{m})}{\Gamma(-j+1+m)\Gamma(-j+1-\bar{m})}\,, \tag{2.5}$$

$$R^{R^\pm}(j,m,\bar{m}) = \nu^{1-2j}\frac{\Gamma(-2j+1)\Gamma(1+\frac{1-2j}{k})}{\Gamma(2j-1)\Gamma(1-\frac{1-2j}{k})}\frac{\Gamma(j+m\pm\frac{1}{2})\Gamma(j-\bar{m}\mp\frac{1}{2})}{\Gamma(-j+1+m\pm\frac{1}{2})\Gamma(-j+1-\bar{m}\mp\frac{1}{2})}\,, \tag{2.6}$$

for primaries of the NS and R sectors, respectively.

The two-dimensional geometry of the axial coset theory is a cigar [16, 17], or Euclidean 2D black hole [18, 19] . It has an asymptotic radius equal to $\sqrt{k\alpha'}$, and the quantum number n is interpreted in the axial $SL(2,\mathbb{R})/U(1)$ coset as the momentum around the compact circle at infinity. The number w denotes the winding number of the closed string states [20], but can be viewed also as the spectral flow parameter of the affine $SL(2,\mathbb{R})$ algebra [21, 22].

The spectrum of the super-coset includes both continuous representations with $j = \frac{1}{2} + iP$ (and $P \in \mathbb{R}_0^+$), and discrete *lowest-weight* representations $\mathcal{D}_j^+$ [20] with values of $j \in \mathbb{R}$ satisfying

$$\frac{1}{2} < j < \frac{k+1}{2}\,, \tag{2.7}$$

and

$$j + r = m\,, \qquad j + \bar{r} = \bar{m}\,, \tag{2.8}$$

where $r, \bar{r}$ are integers (half-integers) for the Neveu-Schwarz (Ramond) sector. The bound for j (2.7) is stricter than the unitarity bound on the coset representations [23] as well as than the bound corresponding to normalizable operators [22]. The improved bound has been shown to apply in all physical settings [5, 22, 24, 25, 26, 8]. Although coset primaries for discrete representations appear for all values of $r, \bar{r}$, only those with $r, \bar{r} \geq 0$ correspond to coset primaries inherited from $SL(2,\mathbb{R})$ primaries, and only for them expressions (2.2) hold. The coset primaries with $r, \bar{r} < 0$ arise from *descendants* of $SL(2,\mathbb{R})$ discrete *lowest-weight* representations $\mathcal{D}_j^+$, but they can be interpreted as *primaries* coming from discrete *highest-weight* representations $\mathcal{D}^-_{\frac{k+2}{2}-j}$ with spin $\frac{k+2}{2} - j$. We discuss the details of the $r, \bar{r} < 0$ primaries in this section, in section 4.1 and in Appendix A.[1]

The spectrum of primaries that we have reviewed above follows from studying the representation of the affine $SL(2,\mathbb{R})$ algebra and from the analysis of the modular invariant partition function of the model [24, 26, 8]. The reflection coefficients (2.5)-(2.6) are related to the dynamics of the theory, i.e. the correlation functions, which we consider now.

2.2 Towards the duality with $N = 2$ Liouville: a bosonic ancestor

Before discussing the duality between the susy coset $SL(2,\mathbb{R})/U(1)$ and $N = 2$ Liouville, we will look at its bosonic counterpart. The correlators of the bosonic $SL(2,\mathbb{R})/U(1)$ theory can be obtained by free field computations in $SL(2,\mathbb{R})$. For this one can use the Wakimoto free field representation of the algebra[2]

$$\begin{aligned} j^+ &= \beta \\ j^3 &= -\beta\gamma - \frac{1}{Q}\partial\phi \\ j^- &= \beta\gamma^2 + \frac{2}{Q}\gamma\partial\phi + k\partial\gamma \end{aligned} \tag{2.9}$$

[1] Note that when we take into account all primaries including those with $r, r' < 0$, there is no need to consider both $\mathcal{D}^+$ and $\mathcal{D}^-$ representations, the $\mathcal{D}^+$ representations being enough to cover all the spectrum.

[2] We will take the $SL(2,\mathbb{R})$ level $k+2$, which is more convenient in order to move to the susy case later. We take $\alpha' = 2$.

where

$$
\begin{aligned}
Q &= \sqrt{\frac{2}{k}} \\
\Delta(\beta,\gamma) &= (1,0) \\
\beta(z)\gamma(w) &\sim \frac{1}{z-w} \\
\phi(w)\phi(z) &\sim -\log(z-w)
\end{aligned}
\tag{2.10}
$$

and the energy momentum tensor of the theory is:

$$
T = \beta\partial\gamma - \frac{1}{2}(\partial\phi)^2 - \frac{Q}{2}\partial^2\phi\,, \tag{2.11}
$$

with the central charge given by (2.1). The $SL(2,\mathbb{R})$ vertex operators are represented by

$$
\Phi^j_{m,\bar{m}} = \gamma^{j+m-1}\bar{\gamma}^{j+\bar{m}-1}e^{(j-1)Q\phi}\,. \tag{2.12}
$$

Correlators were computed in this formalism first in [27], by using the screening charge[3]

$$
\mathcal{L}_1 = \mu_1\beta\bar{\beta}e^{-Q\phi}\,. \tag{2.13}
$$

The two-point function is

$$
\langle\Phi^j_{m,\bar{m}}\Phi^{j'}_{m,\bar{m}}\rangle = |z_{12}|^{-4\Delta_j}\delta_{n+n'}\delta_{w+w'}\left(\delta(j+j'-1)+R^{NS}(j,m,\bar{m})\delta(j-j')\right) \tag{2.14}
$$

where $R^{NS}(j,m,\bar{m})$ is given in (2.5). Similar free field computations can be performed using a dual screening charge [28, 29, 30]

$$
\mathcal{L}_2 = \mu_2(\beta\bar{\beta})^k e^{-\frac{2}{Q}\phi}\,, \tag{2.15}
$$

and the correlators agree under the identification [29]

$$
\pi\mu_2\frac{\Gamma(k)}{\Gamma(1-k)} = \left(\pi\mu_1\frac{\Gamma(k^{-1})}{\Gamma(1-k^{-1})}\right)^k\,. \tag{2.16}
$$

The vertex operators (2.12) correspond to the $SL(2,\mathbb{R})$ theory. To obtain the primaries of the coset one multiplies them by the exponential of a free boson, which represents the gauged coordinate. But the nontrivial part of any correlator computed is the same in $SL(2,\mathbb{R})$ or $SL(2,\mathbb{R})/U(1)$ since both interaction terms $\mathcal{L}_1$ and $\mathcal{L}_2$ commute with the gauged current j^3.

The free field formalism actually allows to compute (2.14) by inserting both $\mathcal{L}_1$ and $\mathcal{L}_2$, and only for those values of j where the anomalous momentum conservation for ϕ is satisfied as

$$
2(j-1)Q - n_1Q - n_2\frac{2}{Q} = -Q\,, \tag{2.17}
$$

[3]Whenever we say screening charge we imply the integrated form $\int d^2z\ \mathcal{L}$.

where $n_1, n_2 = 0, 1, \ldots$ are the number of insertions of $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively[4]. The results are then analytically continued to arbitrary j.

In the limit $\phi \to \infty$ the potential drops exponentially and the theory becomes weakly coupled. In the cigar geometry, the dilaton becomes linear in ϕ, which has the interpretation of the radial coordinate away from the tip. Following [15], it is easy to see that the relation between $\mathcal{L}_1$ and $\mathcal{L}_2$ is that of a strong-weak coupling duality. For $k \to \infty$, $\mathcal{L}_2$ is supported in the strong coupling region ($\phi \to -\infty$) and $\mathcal{L}_1$ has support in the weak coupling region. This is consistent with the fact the $\mathcal{L}_1$ screening can be obtained from the geometry of the parent theory, by parameterizing AdS_3 with Poincare coordinates, and the geometry of the cigar becomes weakly curved at $k \to \infty$ since the radius tends to infinity. The relation is inverted at the opposite limit $k \to 0$. Notice that all this is consistent with the fact that μ_1 and μ_2 are interchanged under $k \leftrightarrow k^{-1}$ as follows from (2.16).

It was first observed in [4] that correlators computed in the bosonic $SL(2,\mathbb{R})/U(1)$ model coincide with those of the sine-Liouville model, whose fields are a compact boson X at the radius $R = \sqrt{2(k+2)}$ of the cigar, and a non-compact one ϕ with background charge. The stress tensor of the theory is

$$T = -\frac{1}{2}(\partial X)^2 - \frac{1}{2}(\partial\phi)^2 - \frac{Q}{2}\partial^2\phi\,, \tag{2.18}$$

and the central charge is[5] $c = 2 + \frac{6}{k}$. The $SL(2,\mathbb{R})/U(1)$ primaries are mapped to the following primaries of sine-Liouville:

$$\Phi^j_{m,\bar{m}} = e^{im\sqrt{\frac{2}{k+2}}X_L + i\bar{m}\sqrt{\frac{2}{k+2}}X_R} e^{(j-1)Q\phi}\,. \tag{2.19}$$

The correlation functions are computed in the Coulomb formalism by using the screening charges

$$\mathcal{L}^{\pm}_{sl} = \mu_{sl} e^{\pm i\sqrt{\frac{k+2}{2}}(X_L - X_R)} e^{-\frac{1}{Q}\rho}\,, \tag{2.20}$$

which are primaries of ± 1 winding in the compact boson. The relevant computations can be found in [31] for the two-point functions and in [32] for some three point functions.[6] Using $\mathcal{L}^{\pm}_{sl}$ as screening charges, the anomalous momentum conservation for a two-point function is

$$2(j-1)Q - (n^- + n^+)\frac{1}{Q} = -Q\,, \tag{2.21}$$

where $n^{\pm}$ is the number of insertions of $\mathcal{L}^{\pm}_{sl}$.

An important result of [4] is that in the computation of N-point functions, with $N \geqslant 3$, the correlators can violate winding number by up to $N-2$ units. This result is easily obtained in the sine-Liouville side, as shown in [32], since the integrals to which the correlators

[4] Note that the values of j selected by (2.17) are nothing but $2j = 1 + n_1 + n_2 k$, which correspond to degenerate representations of the affine $SL(2,\mathbb{R})$ algebra.

[5] This central charge differs by 1 from the central charge in (2.1). This corresponds to the addition of a trivial boson as mentioned before.

[6] The analytical structure of correlators computed in [32], was recently shown in [33] to agree with that obtained in the $SL(2,\mathbb{R})$ approach for winding-violating processes.

reduce in the Coulomb formalism, vanish when the difference $n^- - n^+$ does not have the correct value. The same result, of course, appears in the $SL(2,\mathbb{R})/U(1)$ side, though through some hard work [29, 34]. Note that as a consequence, the perturbative computation of a two-point function requires $n^- = n^+$, and thus an even amount of insertions of $\mathcal{L}_{sl}^{\pm}$.

In terms of strong-weak coupling duality, the sine-Liouville $\mathcal{L}_{sl}^{\pm}$ interaction belongs to the same side of the duality as the interaction Lagrangian $\mathcal{L}_2$. The relation between the coupling constants μ_{sl} and μ_1 was shown in [35] to be

$$\left(\frac{\pi\mu_{sl}}{k}\right)^{\frac{2}{k}} = \pi\mu_1 \frac{\Gamma(k^{-1})}{\Gamma(1-k^{-1})} \tag{2.22}$$

from which, using (2.16) it follows that

$$\left(\frac{\pi\mu_{sl}}{k}\right)^{2} = \pi\mu_2 \frac{\Gamma(k)}{\Gamma(1-k)}. \tag{2.23}$$

This quadratic relation between μ_2 and μ_{sl} is consistent with KPZ scaling, since it is clear from (2.17) that any value of j which is screened with n_2 insertions of $\mathcal{L}_2$, can be equally screened with twice as much insertions of $\mathcal{L}_{sl}^{\pm}$.

2.3 The supersymmetric case

The supersymmetric version of the equivalence between the bosonic coset $SL(2,\mathbb{R})/U(1)$ and the sine-Liouville theory, is the celebrated mirror duality between the supersymmetric coset $SL(2,\mathbb{R})/U(1)$ and the $N=2$ Liouville theory. This duality was conjectured in [5]. It was then shown in [6] that the equivalence between the actions of this two $N=2$ theories follows from mirror symmetry, using the techniques of [36]. The same equivalence was shown in [7] to follow from an analysis of the dynamics of domain walls.

In both analyses of [6] and [7], crucial use is made of the explicit $N=2$ supersymmetry in the action. On the other hand, it is easy to see that the computational content of the duality, i.e., the identity of the correlators, is exactly the same as that of the bosonic version. In the supersymmetric coset $SL(2,\mathbb{R})/U(1)$ side, the primaries are obtained from the parent susy $SL(2,\mathbb{R})_k$ model. In the latter, one can decouple the fermions and shift the level $k \to k+2$ (see Appendix B), so that the NS primary states are the product of a bosonic $SL(2,\mathbb{R})_{k+2}$ primary and the fermionic vacuum. Descending to the coset involves removing a free $U(1)$ boson, so the correlators reduce to those of a purely bosonic model at level $k+2$, with the screening charges given by (2.13) and (2.15). In the Ramond sector, a coset primary is obtained by extracting the contribution of the gauged $U(1)$ from the product of an $SL(2,\mathbb{R})_{k+2}$ primary and a spin field of the decoupled fermions (see e.g. [13]). The computation reduces also to that of the bosonic case, but the dependence on the quantum number m is now shifted to $m \pm \frac{1}{2}$, as can be seen in (2.6).

The $N=2$ Liouville is an interacting theory for a complex chiral super-field Φ, with a Liouville potential. For some previous works on $N=2$ Liouville see [1, 2, 37, 38]. The super-field Φ has a non-compact real component ϕ with background charge, a compact imaginary component Y, and the corresponding fermions. The stress tensor is

$$T = -\frac{1}{2}(\partial Y)^2 - \frac{1}{2}(\partial\phi)^2 - \frac{Q}{2}\partial^2\phi - \frac{1}{2}\psi_{\rm Y}\partial\psi_{\rm Y} - \frac{1}{2}\psi_\phi\partial\psi_\phi \tag{2.24}$$

with a central charge given by (2.1). The compactness of Y actually allows to define the the following chiral/anti-chiral superfields in N=(2,2) superspace of coordinates $(z, \theta^{\pm}; \bar{z}, \bar{\theta}^{\pm})$:

$$\Phi^{\pm} = \phi \pm i(Y_L - Y_R) + i\theta^{\pm}(\tilde{\psi}_\phi \mp i\tilde{\psi}_{\rm Y}) - i\bar{\theta}^{\pm}(\psi_\phi \pm i\psi_{\rm Y}) + i\theta^{\pm}\bar{\theta}^{\pm}F^{\pm}\,. \tag{2.25}$$

Correspondingly we have the following two $N=2$ Liouville chiral interactions:

$$\begin{aligned}\mathcal{L}^{\pm}_{N=2} &= \mu_2 \int d^2\theta^{\pm} e^{-\frac{1}{Q}\Phi^{\pm}} \\ &= \frac{\mu_2}{Q^2} e^{-\frac{1}{Q}[\phi \pm i(Y_L - Y_R)]}(\psi_\phi \pm i\psi_{\rm Y})(\tilde{\psi}_\phi \mp i\tilde{\psi}_{\rm Y})\,.\end{aligned} \tag{2.26}$$

In the expansion we have set to zero the auxiliary field $F^{\pm}$. Its presence only contributes contact terms in the correlators, so it can be ignored [39]. At this point it is convenient to bosonize the fermions as

$$\begin{aligned}\frac{\psi_\phi \pm i\psi_{\rm Y}}{\sqrt{2}} &= e^{\pm iH_L} \\ \frac{\tilde{\psi}_\phi \pm i\tilde{\psi}_{\rm Y}}{\sqrt{2}} &= e^{\pm iH_R}\end{aligned} \tag{2.27}$$

so that the interaction terms become[7]

$$\mathcal{L}^{\pm}_{N=2} = k\mu_2\, e^{-\frac{1}{Q}\phi} e^{\pm i[\frac{1}{Q}Y_L + H_L]} e^{\mp i[\frac{1}{Q}Y_R + H_R]}\,. \tag{2.28}$$

We will now rotate the two bosons Y_L and H_L as

$$\begin{aligned}\sqrt{\frac{k+2}{2}}X_L &= \frac{1}{Q}Y_L + H_L \\ \sqrt{\frac{k+2}{2}}Z_L &= -Y_L + \frac{1}{Q}H_L\end{aligned} \tag{2.29}$$

and similarly for X_R, Z_R. The two bosons X_L, Z_L commute and are canonically normalized. Making this change of variables in the interaction (2.28), we see that Z_L completely decouples from the interaction term, and $\mathcal{L}^{\pm}_{N=2}$ becomes identical to the bosonic $\mathcal{L}^{\pm}_{sl}$, as announced. This change of variables is actually the $N=2$ Liouville equivalent of the chiral rotation in susy $SL(2,\mathbb{R})$ that allows to decouple the fermions by shifting the level k to $k+2$. We again arrived to a form of the screening charge without fermions, and the coefficient of the compact boson in the interaction has been shifted from $\frac{1}{Q} = \sqrt{\frac{k}{2}}$ to $\sqrt{\frac{k+2}{2}}$. The NS primaries of the model are given by (2.19). The Ramond sector is treated as in the $SL(2,\mathbb{R})/U(1)$ case with the same result.

By bosonizing the fermions, we have lost explicit $N=2$ supersymmetry at the level of the action, which was so important in the approaches of [6, 7]. This is not uncommon in conformal field theories, where fermions are typically bosonized in order to compute

[7]The interaction (2.28) is the $N=2$ point in a continuous family of theories studied in [31].

correlators[8]. In our case it is a signal that the reason for the identity of the correlators may not lie in the form of the action of the theory (see later).

A dual, non-chiral interaction term is also allowed by the N=2 symmetry of the N=2 Liouville theory :

$$\tilde{\mathcal{L}}_{N=2} = \tilde{\mu}_2 \int d^4\theta \; e^{\frac{Q}{2}(\Phi^+ + \Phi^-)}. \tag{2.30}$$

We now observe [38] that this screening charge coincides with the screening charge of the $SL(2,\mathbb{R})/U(1)$ super-coset, i.e. the supersymmetric equivalent of (2.13). Using the same steps of bosonization and field redefinitions one can reduce this equivalence to the bosonic one already discussed. Thus this circle of ideas clarifies the fact that the $N = 2$ Liouville duality proposed in [37] is nothing but the bosonic duality of [4] discussed in the previous section.

2.4 Bulk versus localized poles and self-duality

Poles in the correlators of our model can be either of "bulk" or "localized" type. Bulk poles correspond to interactions taking place along the infinite direction of the (asymptotic) linear dilaton. On the other hand, localized poles are associated to discrete normalizable states living near the tip of the cigar[9].

In particular, in the two-point function (2.5), the first two gamma functions of the numerator have bulk poles, and the last two gamma functions, with $m, \bar{m}$ dependence, have localized poles. We analyze here the NS two-point function, the R case being similar.

Let us consider first the bulk poles. The first gamma function has single poles at values $2j = 1, 2, \cdots$. From (2.17), we see that these values of j are screened by $n_1 = 2j - 1$ charges of type $\mathcal{L}_1$, and no $\mathcal{L}_2$ charges. In the Coulomb formalism, one first separates the non-compact field into $\phi = \phi_0 + \tilde{\phi}$, and then the integral of the zero mode ϕ_0 over its infinite volume gives the pole [41]. The second gamma function has poles at $\frac{(2j-1)}{k} = 1, 2....$ These values of j are screened in $SL(2,\mathbb{R})/U(1)$ by $n_2 = \frac{(2j-1)}{k}$ charges of type $\mathcal{L}_2$, and no $\mathcal{L}_1$ charges. In the sine-Liouville theory, these corresponds to having $n^+_{sl} = \frac{(2j-1)}{k}$ charges of $\mathcal{L}^+_{sl}$, *and* the same amount of $\mathcal{L}^-_{sl}$. This follows from (2.21) and the condition $n^-_{sl} = n^+_{sl}$ for two-point functions.

The same phenomenon of having two families of bulk poles occurs in bosonic Liouville theory, where the two-point function has two sets of poles, their semi-classical origin being the insertion of either the Liouville interaction $\mu_L e^{-2b\phi}$ or its dual $\tilde{\mu}_L e^{-\frac{2}{b}\phi}$, with a relation between μ_L and $\tilde{\mu}_L$ similar to (2.16) [42].

The bulk poles in (2.5) are simple poles, except at the level $k = 1$. At this level, corresponding to $c = 9$, both the first *and* second gamma functions in (2.5) have each a pole at $2j = 2, 3....$ This is signaled by the fact that the two dual charges $\mathcal{L}_1$ and $\mathcal{L}_2$

[8]Explicit $N = 2$ supersymmetry at the level of the action is not necessary for a conformal field theory to have $N = 2$ supersymmetry in its spectrum and in its chiral algebra. For example, an $N = 2$ minimal model with central charge $c = 1$ can be realized through a free compact boson. We thank A.Giveon for comments on this point.

[9]See [40] for a recent discussion on this double nature of poles in the context of holographic descriptions of Little String Theories.

become equal for $k = 1$, including $\mu_1 = \mu_2$, so this the self-dual point of the theory. There is only one single bulk pole left at $k = 1$, that comes from the first gamma function of (2.5) at $2j = 1$. But this is the expected result since for this j no screening charges are needed to satisfy the anomalous momentum conservation (2.17), and the single pole comes from the infinite volume of the zero mode of ϕ.[10]

Notice that although one can interpret $\mathcal{L}_1$ as being a strong-weak dual to both $\mathcal{L}_2$ and $\mathcal{L}_{sl}^{\pm}$, the sine-Liouville interaction remains outside the liaison between the Wakimoto pair $\mathcal{L}_1$ and $\mathcal{L}_2$, and this becomes more manifest at the self-dual point $k = 1$.

As mentioned, localized poles occur in the third and fourth gamma functions of the numerator of (2.5). They signal the presence of normalizable bound states in the strong coupling region associated with the residue of the singularity. We will consider here the case $m = \bar{m} = kw/2$, which is the relevant case for the D-brane analysis. At values of $r = kw/2 - j \geqslant 0$, corresponding to primaries of $SL(2,\mathbb{R})/U(1)$ coming from primaries of $SL(2,\mathbb{R})$, the fourth gamma function in the numerator of (2.5) has a localized pole. On the other hand, at values of $r = kw/2 - j < 0$ corresponding to primaries of $SL(2,\mathbb{R})/U(1)$ which are descendants of $SL(2,\mathbb{R})$, the two-point function has a zero from a pole in the third gamma function in the denominator of (2.5). We will return to this issue in sect. 4.1.

A qualitative feature that we wish to stress is that, for the theories with boundary that we will construct, *the two kinds of poles naturally decouple.* One-point functions of localized D0-branes will have only poles associated to the discrete normalizable states, and those of extended D1-branes will have only "bulk" poles associated to the zero-mode of the radial coordinate. Intuitively, on the one hand, the D0-branes are localized near the tip of the cigar, as are the normalizable bound states, and on the other hand, the D1-branes stretch along the radial direction and only couple to momentum modes, thus forbidding the coupling to discrete bound states that all carry a non-trivial winding charge. In the last case of D2-branes, we have both type of poles since these non-compact branes have a induced D0-brane charge localized at the tip of the cigar [43].

A similar decoupling phenomenon can be observed for the one-point functions of bosonic (and $N = 1$ [44]) Liouville theory. In this case, the ZZ one-point functions [45] have no poles and the FZZT one-point functions [46] have the bulk Liouville poles mentioned above.

2.5 The conformal bootstrap approach

We have reviewed above how perturbative calculations with different screening charges lead to the same correlators in both theories. The perturbative approach leads to physical insights related to the nature of the poles, strong-weak coupling regimes, etc. Also, in the last years a new powerful approach to compute correlators in non-rational conformal field theories has appeared [47]. In non-rational conformal field theories the normalizable states have a continuous spectrum (in H_3^+ they correspond to the continuous representations of $SL(2,\mathbb{R})$) and appear in the intermediate channels of the correlators. The new approach

[10] The importance of the self-dual point at $k = 1$ has recently been stressed in [33]. The same self-duality phenomenon, with single poles becoming double poles, occurs in Liouville at $c = 25$ ($b = 1$).

consists in assuming that a general property of the conformal bootstrap, namely the factorization constraints, can be analytically continued to primary states corresponding to non-normalizable degenerate operators with discrete spectrum. This assumption, together with assuming that a strong-weak coupling duality is present in the theory (of the type between $\mathcal{L}_1$ and $\mathcal{L}_2$ above), leads to constraints for two and three point functions which have a unique solution. We will not review this method here, and we refer the reader to [47, 42, 46] for details.

A natural question is what new light can be shed on the equivalence between $N = 2$ Liouville and the susy $SL(2,\mathbb{R})/U(1)$ through these methods. This question is related to the more general issue of what role the action or the perturbative screening charges play in this approach. The method essentially reduces to a minimum the dynamical information needed to solve the theory. It asks as an input two pieces of information: *i)* the quantum numbers of a degenerate operator, which are given by the chiral symmetries of the theory, and *ii)* the value of certain fixed correlators involving one or two screenings. As an output we get the value of arbitrary correlators. In the sine-Liouville/ $SL(2,\mathbb{R})/U(1)$ context, this method has been exploited in [35] to obtain the relation (2.22) between coupling constants μ_1 and μ_{sl}.

Now, as shown in [47], it turns out that the second piece of input, namely, the perturbative fixed correlators, can be obtained by asking for consistency of the factorization constraints themselves[11]. This means that the chiral algebra of the model would fully fix the correlators, through the family of its degenerate primaries. The argument holds both for bulk and boundary theories. In this way, for example, in [49], factorization constraints for the boundary H_3^+ theory were obtained without introducing a boundary action. In other words, under certain analyticity assumptions one could in principle achieve for non-rational conformal field theories, what is known to hold for rational ones, namely, that the chiral symmetries of the theory completely fix the correlation functions (under a certain prescription as to how left and right fields are glued). Notice that in our theory many factors of the two-point function (2.5) can be seen as the result of Fourier-transforming the same object from a basis of primaries where the $SL(2,\mathbb{R})$ chiral symmetry is realized through differential operators (see Appendix B). So the idea would be to push these symmetry constraints further to their very end.

Carrying this program in our case, the $SL(2,\mathbb{R})/U(1)$ and $N = 2$ Liouville theories would appear just as different realizations of the same chiral structure. The latter would be nothing but the common core of their respective chiral algebras, affine $SL(2,\mathbb{R})$ and $N = 2, c > 3$ Virasoro, which are [23] related by a free boson which does not affect the correlators. In appendix D we show how an $N = 2, c > 3$ algebra yields always an $SL(2,\mathbb{R})$ algebra by adding a free boson. Concerning the possible additional "geometrical" information, namely, the way left and right chiral fields are glued, there is only one known modular invariant with $N = 2, c > 3$ spectrum [26, 8], up to discrete orbifolds acting on the N=2 charges.

That is the idea behind a central assumption of our paper, namely, that boundary

[11] We thank V. Schomerus for this crucial comment.

CFT quantities computed in H_3^+, which descend naturally to $SL(2,\mathbb{R})/U(1)$, describe also the boundary theory of $N=2$ Liouville.

3. The boundary

The bosonic coset

Recall that for the bosonic coset we have that the one-point functions [43] are given in terms of the product of one-point functions for H_3^+ and the one-point functions for an auxiliary boson X (at radius $R=\sqrt{\alpha' k}$) which we can give the geometric interpretation of being the angular variable in the coset. The left and right chiral $U(1)$ quantum numbers of H_3^+ (labeled n_H and p_H) and X are related as:

$$\left(\frac{n_H+ip_H}{2},-\frac{n_H-ip_H}{2}\right)=\left(\frac{n+kw}{2},\frac{n-kw}{2}\right) \tag{3.1}$$

where the relative minus sign arises because we gauge axially. We then use the factorization of the one-point function:

$$\langle\Phi^{j,coset}_{n,w}\rangle=\langle\Phi^{j,H_3}_{n,w}\rangle\langle V^X_{n,w}\rangle \tag{3.2}$$

to obtain the one-point function in the coset [43]. Since X has the geometric interpretation of being the angular variable, Dirichlet conditions on X (i.e. Neumann conditions on the H_3^+ gauged current) have the interpretation of branes localized in the angular direction.[12] Note that we have assumed the ghost contribution to the one-point function to be trivial. (We can detect non-trivial renormalizations through the Cardy check.) Note also that our derivation is only valid for coset primaries that are associated to *primaries* in the parent theory. Coset primaries associated to descendents in the original model require special care.

The super-coset

For the super-coset, we can tell an analogous story. We have that the one-point functions are given by a product of one-point functions for H_3^+ at level $k+2$, for a $U(1)$ associated to the fermions, and for an auxiliary boson X that has again the interpretation of the angular direction. The one-point functions factorize, and we have:

$$\langle\Phi^{j,supercoset}_{n,w}\rangle=\langle\Phi^{j,H_3}_{n,w}\rangle\langle V^X_{n,w}\rangle\langle V_F\rangle. \tag{3.3}$$

We can be more precise about the relationship between the boundary condition for the various currents. The left and right $N=2$ R-currents are given in terms of the total currents J^3, $\bar{J}^3$ as follows (see appendix A for conventions) :

$$J^R=\psi^+\psi^-+\frac{2J^3}{k}\ \text{and}\quad \bar{J}^R=\tilde{\psi}^+\tilde{\psi}^--\frac{2\bar{J}^3}{k}. \tag{3.4}$$

[12]See next paragraph for a more rigorous argument based on BRST symmetry.

The *A-type* boundary conditions of the N=2 algebra [50] are defined through the *twisted* gluing conditions[13]: $J^R = \bar{J}^R$, $G^{\pm} = i\eta\tilde{G}^{\mp}$, $\eta = \pm 1$ being the choice of spin structure. Thus we see from (3.4) – and the expressions for the supercurrents – that the total currents of the $SL(2,\mathbb{R})$ algebra has to satisfy *untwisted* boundary conditions: $J^{3,\pm} = -\bar{J}^{3,\pm}$. As we know from [51, 48, 49] this corresponds to AdS_2 branes in AdS_3. The axial super-coset has a BRST charge corresponding to the gauge-fixed local symmetry, whose expression is (see e.g. [52]) :

$$\mathcal{Q}_{BRST} = \oint \frac{dz}{2i\pi} \left\{ c\,(J^3 + i\partial X) + \gamma\,(\psi^3 + \psi^{\mathrm{X}}) \right\} - \oint \frac{d\bar{z}}{2i\pi} \left\{ \tilde{c}\,(\bar{J}^3 - i\bar{\partial} X) + \bar{\gamma}\,(\tilde{\psi}^3 - \tilde{\psi}^{\mathrm{X}}) \right\}. \tag{3.5}$$

Thus the preservation of the BRST current will impose that the extra boson X has the boundary conditions $\partial X = \bar{\partial} X$ in the closed string channel, i.e. Dirichlet conditions. The net effect of the (β, γ) super-ghosts will be to remove the contributions of the fermions ψ^3 associated to J^3 and ψ^{X} associated to X, leaving the fermions $\psi^{\pm}$ with (relative) A-type boundary conditions. In the cigar these are the D1-branes, extending to infinity [53, 43].

The *B-type* boundary conditions of the N=2 algebra [50] are defined through the *untwisted* gluing conditions: $J^R = -\bar{J}^R$, $G^{\pm} = i\eta\tilde{G}^{\pm}$. Using the same lines of reasoning we find that these boundary conditions corresponds to *twisted* boundary conditions for the $SL(2,\mathbb{R})$ currents, $J^3 = \bar{J}^3, J^{\pm} = -\bar{J}^{\mp}$. These are either H_2 branes or S^2 of imaginary radius. In the former case we obtain D2-branes in the cigar, and in the latter D0-branes localized at the tip of the cigar [53, 43]. In both cases the extra field X has to satisfy Neumann boundary conditions.

To summarize, the one-point functions for the super-coset are as in [43] (but, importantly, the basis of Ishibashi states to which they correspond is a basis of Ishibashi states that preserves $N = 2$ superconformal symmetry and the level is shifted by two units), with an additional factor corresponding to two real fermions.

We move on to apply the dictionary above to the particular cases of D0-, D1- and D2-branes in the supersymmetric $N = 2$ theories. We will be using the quantum number notations traditional for the $SL(2,\mathbb{R})/U(1)$ super-coset, for convenience of comparison with (technically similar) results in the bosonic coset [43]. But it should always be kept in mind that the construction equally well applies to $N = 2$ Liouville theory, since the primaries in one theory can be associated to unique primaries in its dual and since the characters in both theories are identical.

4. D0-branes

The first type of branes we discuss are the B-type branes localized at the tip of the cigar $SL(2,\mathbb{R})/U(1)$ conformal field theory.

4.1 One-point function for the localized branes

These one point functions for the D0-branes of N=2 Liouville are similar to those of the

[13] All the gluing conditions in this section correspond to the closed string channel.

ZZ branes [45] for bosonic Liouville theory. Their expression is (see Appendix B) :

$$\langle \Phi^j_{nw} \rangle^{D0}_u = \delta_{n,0} \frac{\Psi_u(j,w)}{|z-\bar{z}|^{\Delta_{j,w}}} \tag{4.1}$$

where

$$\begin{aligned}
\Psi^{NS}_u(j,w) &= k^{-\frac{1}{2}}(-1)^{uw}\nu^{\frac{1}{2}-j} \frac{\Gamma(j+\frac{kw}{2})\Gamma(j-\frac{kw}{2})}{\Gamma(2j-1)\Gamma(1-\frac{1-2j}{k})} \frac{\sin\frac{\pi}{k}u(2j-1)}{\sin\frac{\pi}{k}(2j-1)} \\
\Psi^{\widetilde{NS}}_u(j,w) &= i^w \Psi^{NS}_u(j,w) \\
\Psi^{R\pm}_u(j,w) &= k^{-\frac{1}{2}}(-1)^{uw}\nu^{\frac{1}{2}-j} \frac{\Gamma(j+\frac{kw}{2}\pm\frac{1}{2})\Gamma(j-\frac{kw}{2}\mp\frac{1}{2})}{\Gamma(2j-1)\Gamma(1-\frac{1-2j}{k})} \frac{\sin\frac{\pi}{k}u(2j-1)}{\sin\frac{\pi}{k}(2j-1)}
\end{aligned} \tag{4.2}$$

The normalization constant $k^{-\frac{1}{2}}(-1)^{uw}$ (and i^w) is fixed to satisfy the Cardy condition (see next section).

The numbers $u = 1, 2...$ are inherited from the one-point functions of localized D-branes in H_3^+, and they correspond to finite dimensional representations of $SL(2,\mathbb{R})$ of spin $j = -\frac{(u-1)}{2}$. The latter are only a subset of the degenerate representations of the $SL(2,\mathbb{R})$ affine algebra[14]. Remember that in the bosonic and $N = 1$ Liouville theory, the localized ZZ branes [45, 44] have *two* quantum numbers, associated to all the degenerate representations of the ($N = 1$) Virasoro algebra. This suggests that more general solutions of the factorization constraints in [49] are expected, which in turn would imply a bigger family of D-branes for our $N = 2$ model.

For a discrete state with pure winding, the numbers j and w are correlated as [8]

$$j + r = \frac{kw}{2} \tag{4.3}$$

with $r \in \mathbb{Z}$ for the NS sector and $r \in \mathbb{Z}+\frac{1}{2}$ for the R sector. Moreover, the unitarity bound (2.7) for j implies that for every r there is a unique pair of j, w such that (4.3) holds.

Given a j in the unitary bound (2.7) and satisfying (4.3), the one point functions (4.2) have poles at values of $r \geq 0, w > 0$. The case $r < 0$ should be treated differently, since from the point of view of the parent H_3^+ – or $SL(2,\mathbb{R})$ – theory, the states with $w > 0$ and $w \leqslant 0$ are of very different origin. Indeed in the former case, the equation (4.3) can be solved with $r \geqslant 0$, hence those states descend from flowed primaries of a lowest weight representation $\mathcal{D}^{j,w}_+$ of the $SL(2,\mathbb{R})$ affine algebra, see [21, 22]. Accordingly the formulae (4.2) are obtained from descent of the H_3^+ ones – as in [43] – valid for primaries of H_3^+. The one-point function has simple poles for all these states, coming form the second Gamma function in the numerator of (4.2). On the contrary, in the latter case $w \leqslant 0$, the solution of (4.3) is solved with $r < 0$. Those states, while primaries of the coset,[15] are *descendents* of the $SL(2,\mathbb{R})$ flowed algebra. To use nevertheless the formulas of the H_3^+ branes, we have to use the isomorphism of representations : $\mathcal{D}^{+,w}_j \sim \mathcal{D}^{-,w-1}_{j'}$, with $j' = \frac{k+2}{2} - j$. Applying this mapping on the one-point functions, we find poles coming form the first Gamma function in the numerators of (4.2).

[14]They correspond to $n_2 = 0$ in footnote 4.

[15]In fact these "diagonal" states are obtained from the lowest weight state of the representation. In a bosonic model they are: $(J^-_{-1})^r|j,j\rangle$. For the supersymmetric case, see appendix A.

4.2 Cardy computation for the D0-branes

In this section we will verify that the one-point functions of localized D-branes obtained in section 4.1. satisfy the Cardy condition relating the open and closed string channels. We distinguish between three cases for the open string spectrum: NS, R and $\widetilde{\text{NS}}$. As usual [54], they correspond to NS, $\widetilde{\text{NS}}$ and R sectors in the closed string channel, respectively. We will show below the computation in detail for the NS/NS case. For the other cases, we state the result, and defer details to Appendix C.

For the annulus partition function of a NS open string stretching between a brane labeled by $u = 1, 2...$ and the basic brane with $u' = 1$ we consider

$$\begin{aligned} Z^{NS}_{u,1}(\tau,\nu) &= \sum_{r\in\mathbb{Z}} ch^{NS}_f(u,r;\tau,\nu) \\ &= \frac{\vartheta_3(\tau,\nu)}{\eta(\tau)^3} \sum_{s\in\mathbb{Z}+\frac{1}{2}} \frac{1}{1+yq^s}\left(q^{\frac{s^2-su}{k}} y^{\frac{2s-u}{k}} - q^{\frac{s^2+su}{k}} y^{\frac{2s+u}{k}}\right) . \end{aligned} \tag{4.4}$$

We have taken the $N = 2$ NS characters (A.37) associated to the u-dimensional representations of $SL(2,\mathbb{R})$ of spin $j = -\frac{(u-1)}{2}$, and we have summed over the whole spectral flow orbit. For an open string stretching between two branes with general boundary conditions u and u', we expect, as in [55, 49, 43], that the partition function is obtained by summing $Z^{NS}_{u,1}$ over the irreducible representations appearing in the fusion of the representations associated to u and u'. Calling the latter $\mathbf{j} = |j|$ and $\mathbf{j}' = |j'|$, we have

$$\mathbf{j} \otimes \mathbf{j}' = |\mathbf{j} - \mathbf{j}'| \;\oplus\; |\mathbf{j} - \mathbf{j}'| + 1 \oplus \cdots \oplus\; \mathbf{j} + \mathbf{j}' . \tag{4.5}$$

In terms of the u, u' indices, this decomposition implies

$$Z^{NS}_{u,u'}(\tau,\nu) = \sum_{n=0}^{\min(u,u')-1} Z^{NS}_{u+u'-2n-1,1}(\tau,\nu) . \tag{4.6}$$

In order to verify the Cardy condition on these branes, we will perform a modular transformation of $Z^{NS}_{u,u'}$ to the closed string channel[16]. Let us start with the $Z^{NS}_{u,1}$ partition function (4.4). We will parameterize ν as: $\nu = \nu_1 - \tau\nu_2$, $\nu_{1,2} \in \mathbb{R}$. The modular transform of $Z^{NS}_{u,1}(\tau,\nu)$ can be expressed as

$$\begin{aligned} e^{-i\pi\frac{c}{3}\frac{\nu^2}{\tau}} \quad Z^{NS}_{u,1}\left(-\frac{1}{\tau},\frac{\nu}{\tau}\right) &= \frac{\vartheta_3(\tau,\nu)}{\eta(\tau)^3} \\ \times \quad \frac{1}{2i\pi}\left[\int_{\mathcal{C}_{-\epsilon}} + \int_{\mathcal{C}_{+\epsilon}}\right] dZ \; (-i\pi)\, e^{\pi Z + \frac{2i\pi\tau}{k}Z^2} \; &\frac{\sinh(2\pi\frac{Z}{k}u)}{\cosh(\pi Z)} \frac{e^{i\pi(i\tau Z-\nu)}}{\cos\pi(i\tau Z-\nu)} \end{aligned} \tag{4.7}$$

where the parameter ϵ is infinitesimal and positive. The contour encircles the line $\mathcal{C}:\; Z = iy/\tau + i\nu_2$, $y \in \mathbb{R}$ (see fig. 1). This identity is valid because we pick up all the poles of

[16] The result for the NS/NS case follows as a particular case of an identity proved in [56, 57], and the other cases considered below ($\widetilde{\text{NS}}$/R, R/$\widetilde{\text{NS}}$ and $\mathbb{Z}_p$ orbifold) are variations thereof. We perform the computation here for completeness and in order to adapt the notation to our present purposes.

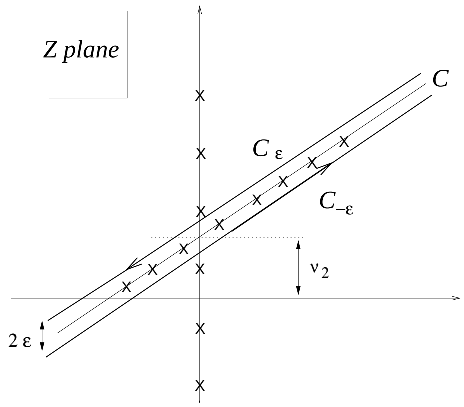

Figure 1: Choice of contour of integration (for $\tau_1 > 0$).

the integrand inside the contour. These are the zeroes of $\cos\pi(i\tau Z - \nu)$ which occur at $\nu - i\tau Z = y + \nu_1 = s \in \mathbb{Z} + 1/2$. At these poles, the contour integral yields the residues

$$-\frac{2}{\tau}\frac{e^{i\pi(s-\nu)/\tau}\ \sin(2\pi\frac{u}{k}(s-\nu)/\tau)\ e^{-i2\pi(s-\nu)^2/(k\tau)}}{e^{i\pi(s-\nu)/\tau}+e^{-i\pi(s-\nu)/\tau}}, \tag{4.8}$$

which lead to the identity (4.7). This identity is valid as long as there are no poles coming from the $\cosh\pi Z$ factor on the contour of integration. These special cases occur only for $\nu_2 \in \mathbb{Z} + 1/2$ and we assume that ν_2 does not take these values. We now note that we have the expansions:

$$\frac{e^{i\pi(i\tau Z-\nu)}}{2\cos\pi(i\tau Z-\nu)} = \sum_{w=0}^{\infty}(-1)^w e^{-2\pi i w(i\tau Z-\nu)} \quad \text{for} \quad |e^{-i2\pi(i\tau Z-\nu)}| < 1 \tag{4.9}$$

and

$$\frac{e^{i\pi(i\tau Z-\nu)}}{2\cos\pi(i\tau Z-\nu)} = -\sum_{w=-\infty}^{-1}(-1)^w e^{-2\pi i w(i\tau Z-\nu)} \quad \text{for} \quad |e^{+i2\pi(i\tau Z-\nu)}| < 1\,. \tag{4.10}$$

The expansion (4.9) is valid in $\mathcal{C}_{+\epsilon}$ and (4.10) is valid in $\mathcal{C}_{-\epsilon}$. Plugging these expansions into the right hand side of eq. (4.7) we get

$$e^{-i\pi\frac{c}{3}\frac{\nu^2}{\tau}} Z^{NS}_{u,1}(-\frac{1}{\tau},\frac{\nu}{\tau}) = \frac{\vartheta_3(\tau,\nu)}{\eta(\tau)^3}\sum_{w\in\mathbb{Z}}\int_{\mathcal{C}} dZ\ (-1)^w\ \frac{\sinh(2\pi\frac{Z}{k}u)}{\cosh(\pi Z)} e^{\pi Z}\ y^w q^{-iwZ+\frac{Z^2}{k}}\,. \tag{4.11}$$

When taking $\epsilon \to 0$ we added a minus sign to the contour $\mathcal{C}_{+\epsilon}$ and switched its direction. To obtain the corresponding expression for the general case (4.6) of $Z^{NS}_{u,u'}$, we use the identity

$$\sum_{n=0}^{\min(u,u')-1}\sinh\left(2\pi\frac{Z}{k}(u+u'-2n-1)\right) = \frac{\sinh(2\pi\frac{Z}{k}u)\sinh(2\pi\frac{Z}{k}u')}{\sinh(2\pi\frac{Z}{k})} \tag{4.12}$$

so we get

$$e^{-i\pi\frac{c}{3}\frac{\nu^2}{\tau}} Z^{NS}_{u,u'}(-\frac{1}{\tau},\frac{\nu}{\tau}) =$$
$$\frac{\vartheta_3(\tau,\nu)}{\eta(\tau)^3}\sum_{w\in\mathbb{Z}}\int_{\mathcal{C}} dZ\ (-1)^w\ \frac{\sinh(2\pi\frac{Z}{k}u)\sinh(2\pi\frac{Z}{k}u')}{\cosh(\pi Z)\sinh(2\pi\frac{Z}{k})} e^{\pi Z}\ y^w q^{-iwZ+\frac{Z^2}{k}}\ . \quad (4.13)$$

Note that the exponent of q is complex. In order to get a real exponent for q we will *(i)* shift the $\mathcal{C}$ contour of integration by $\frac{iwk}{2} - i\nu_2$, for each term indexed by w and *(ii)* tilt it parallel to the real axis (see fig. 2). The resulting exponent of q will be that needed to get the characters of the continuous representations. As we shift the contour we pick poles which

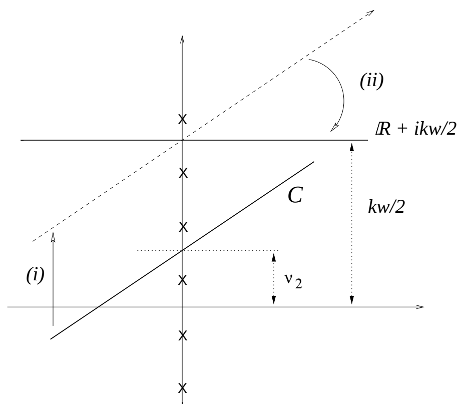

Figure 2: Change of contour of integration (case $w > \frac{k\nu_2}{2}$).

will make up the contributions of the $N = 2$ discrete representations to the closed string channel amplitude. No poles are picked when tilting to the real axis. So the modular transform of $Z^{NS}_{u,u'}$ is decomposed into

$$e^{-i\pi\frac{c}{3}\frac{\nu^2}{\tau}} Z^{NS}_{u,u'}(-\frac{1}{\tau},\frac{\nu}{\tau}) = Z^{NS,c}_{u,u'} + Z^{NS,d}_{u,u'}\,. \quad (4.14)$$

The first term comes from the continuous integral, which after shifting and tilting the contour, becomes

$$Z^{NS,c}_{u,u'} = \sum_{w\in\mathbb{Z}}\int_{-\infty}^{+\infty} dP \frac{e^{\pi(\frac{iwk}{2}+P)}\sinh(2\pi\frac{P}{k}u)\sinh(2\pi\frac{P}{k}u')}{(-1)^{w(u-u')}\cosh(\pi(P+i\frac{wk}{2}))\sinh(2\pi\frac{P}{k})} \quad (4.15)$$
$$\times\ q^{\frac{P^2+(wk/2)^2}{k}} y^w \frac{\vartheta_3(\tau,\nu)}{\eta(\tau)^3}\,.$$

In the last factor we get a continuous character $ch^{NS}_c(P,\frac{wk}{2};\tau,\nu)$ (see (A.20)), corresponding to a pure winding state with $m = \frac{kw}{2}$, as expected. And since $ch_c(P,\frac{wk}{2};\tau,\nu)$ is even in P we can rewrite this as

$$Z^{NS,c}_{u,u'} = \sum_{w\in\mathbb{Z}}\int_0^{+\infty} dP \frac{2\sinh(2\pi P)\sinh(2\pi\frac{P}{k}u)\sinh(2\pi\frac{P}{k}u')}{(-1)^{w(u-u')}[\cosh(2\pi P)+\cos(\pi kw)]\sinh(2\pi\frac{P}{k})} ch^{NS}_c(P,\frac{wk}{2};\tau,\nu)\,. \quad (4.16)$$

This expression is equal to

$$Z_{u,u'}^{NS,c} = \sum_{w\in\mathbb{Z}} \int_0^{+\infty} dP\ \Psi_u\left(\frac{1}{2} - iP, -w\right) \Psi_{u'}\left(\frac{1}{2} + iP, w\right)\ ch_c^{NS}(P, \frac{wk}{2}; \tau, \nu) \quad (4.17)$$

so we get the expected continuous spectrum from the overlap between D0 boundary states[17].

The poles picked up arise from the zeroes of the factor $\cosh \pi Z$ in (4.13), at values of Z such that $Z = is$ with $s \in \mathbb{Z} + \frac{1}{2}$. Note that for irrational k the shifted contour will never fall on a pole, and we will assume this to be the case. The poles will contribute with positive sign for $s > \nu_2$ and with negative sign for $s < \nu_2$. Let us consider the former case. A given $s \in \mathbb{Z} + \frac{1}{2}$ will give a pole contribution for all the terms in (4.11) with $w \geqslant w_s$, where w_s is the only integer satisfying

$$w_s > \frac{2s}{k} > w_s - 1\,. \quad (4.18)$$

For poles corresponding to $s < \nu_2$, we get contributions for all the terms in (4.11) with $w \leqslant w_s - 1$, where w_s is also defined by (4.18). Summing all these residues we get

$$Z_{u,u'}^{NS,d} = -2\frac{\vartheta_3(\tau,\nu)}{\eta(\tau)^3}\left[\sum_{s>\nu_2}\sum_{w=w_s}^{\infty} - \sum_{s<\nu_2}\sum_{w=-\infty}^{w_s-1}\right] \frac{\sin\left(2\pi\frac{s}{k}u\right)\sin\left(2\pi\frac{s}{k}u'\right)}{(-1)^w \sin\left(2\pi\frac{s}{k}\right)}\ y^w q^{sw-\frac{s^2}{k}}\,. \quad (4.19)$$

Noting that $|yq^s| = e^{-2\pi\tau_2(s-\nu_2)}$ is smaller (bigger) than 1 for $s > \nu_2$ ($s < \nu_2$), we can sum on w for every s to get

$$Z_{u,u'}^{NS,d} = -2\sum_{s\in\mathbb{Z}+\frac{1}{2}} (-1)^{w_s} \frac{\sin\left(2\pi\frac{s}{k}u\right)\sin\left(2\pi\frac{s}{k}u'\right)}{\sin\left(2\pi\frac{s}{k}\right)}\ \frac{y^{w_s}q^{sw_s-\frac{s^2}{k}}}{1+yq^s}\frac{\vartheta_3(\tau,\nu)}{\eta(\tau)^3}\,. \quad (4.20)$$

This result can be recast in a more transparent fashion as follows. Calling $s = r + \frac{1}{2}$ with $r \in \mathbb{Z}$, a character of the $N = 2$ NS discrete representations corresponding to a pure winding mode $J_0^3 = j + r = \frac{kw}{2}$ is given by (see (A.31))

$$ch_d^{NS}(j, r; \tau, \nu) = \frac{y^w q^{sw-\frac{s^2}{k}}}{1+yq^s}\frac{\vartheta_3(\tau,\nu)}{\eta(\tau)^3}\,. \quad (4.21)$$

Now, for every $r \in \mathbb{Z}$ there is *only one* value of w, such that $j = -r + \frac{kw}{2}$ lies in the improved unitary bound (2.7). This is exactly the value of w fixed by the condition (4.18). Moreover, *all* possible representations corresponding to pure winding states and such that j lies inside the unitary bound are covered by taking all $r \in \mathbb{Z}$ and fixing w_s as in (4.18). Calling j_r the spin of the representation associated to each r, then (4.20) is equal to

$$Z_{u,u'}^{NS,d} = \sum_{r\in\mathbb{Z}} (-1)^{w_r(u-u')} \frac{2\sin\left(\frac{\pi}{k}(2j_r-1)u\right)\sin\left(\frac{\pi}{k}(2j_r-1)u'\right)}{\sin\left(\frac{\pi}{k}(2j_r-1)\right)}\ ch_d^{NS}(j_r, r; \tau, \nu) \quad (4.22)$$

[17] Note that in the "out" boundary state we take the opposite $U(1)$ charge, as follows from CPT conjugation [58].

where w_s has been renamed w_r.

As shown in section 4.1, the product $\Psi_u(-j+1,-w)\Psi_{u'}(j,w)$ has a single pole for every discrete, pure-winding state. It is natural to expect the discrete part of the annulus amplitude (in the closed string channel) to be given by the residue of this pole. This is indeed the case, and one can check that (4.22) is equal to

$$Z^{NS,d}_{u,u'} = 2\pi \sum_{r\in\mathbb{Z}} Res\left[\Psi^{NS}_u(-j_r+1,-w_r)\Psi^{NS}_{u'}(j_r,w_r)\right]\ ch^{NS}_d(j_r,r;\tau,\nu)\,, \tag{4.23}$$

where the residue is computed when considering the bracketed expression as an analytic function of j. More details about this last step are given in section 7 that treats D2-branes. We have thus verified the Cardy consistency condition for the D0 branes in the NS/NS sector.

The computation for the R/$\widetilde{\text{NS}}$ and $\widetilde{\text{NS}}$/R cases is basically the same, *mutatis mutandi.* We state here the results and provide the details of the computation in Appendix C for the interested reader. For the open string sector we take the partition functions

$$Z^{R}_{u,1}(\tau,\nu) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} ch^R_f(u,r;\tau,\nu)\,, \tag{4.24}$$

$$Z^{\widetilde{NS}}_{u,1}(\tau,\nu) = \sum_{r\in\mathbb{Z}} ch^{\widetilde{NS}}_f(u,r;\tau,\nu)\,, \tag{4.25}$$

and $Z^{R/\widetilde{NS}}_{u,u'}(\tau,\nu)$ is given by a sum as in (4.6). The modular transforms are

$$\begin{aligned}
&e^{-i\pi\frac{c}{3}\frac{\nu^2}{\tau}} Z^R_{u,u'}(-\frac{1}{\tau},\frac{\nu}{\tau}) = \\
&\sum_{w\in\mathbb{Z}}\int_0^{+\infty} dP\ \Psi^{\widetilde{NS}}_u\left(\frac{1}{2}-iP,-w\right)\Psi^{\widetilde{NS}}_{u'}\left(\frac{1}{2}+iP,w\right)\ ch^{\widetilde{NS}}_c(P,\frac{wk}{2};\tau,\nu) \\
&\quad +2\pi\sum_{r\in\mathbb{Z}} Res\left[\Psi^{\widetilde{NS}}_u(-j_r+1,-w_r)\Psi^{\widetilde{NS}}_{u'}(j_r,w_r)\right]\ ch^{\widetilde{NS}}_d(j_r,r;\tau,\nu)\,,
\end{aligned} \tag{4.26}$$

$$\begin{aligned}
&e^{-i\pi\frac{c}{3}\frac{\nu^2}{\tau}} Z^{\widetilde{NS}}_{u,u'}(-\frac{1}{\tau},\frac{\nu}{\tau}) = \\
&\sum_{w\in\mathbb{Z}}\int_0^{+\infty} dP\ \Psi^R_u\left(\frac{1}{2}-iP,-w\right)\Psi^R_{u'}\left(\frac{1}{2}+iP,w\right)\ ch^R_c(P,\frac{wk}{2};\tau,\nu) \\
&\quad +2\pi\sum_{r\in\mathbb{Z}+\frac{1}{2}} Res\left[\Psi^R_u(-j_r+1,-w_r)\Psi^R_{u'}(j_r,w_r)\right]\ ch^R_d(j_r,r;\tau,\nu)\,,
\end{aligned} \tag{4.27}$$

where j_r and w_r are defined in the same way as for the NS/NS case, but notice that in the the R case r is half-integer. In both (4.26) and (4.27) the residues are again computed considering the bracketed expression as a function of j.

Some comments are in order.

- Although the Cardy condition holds for arbitrary boundary conditions u, u', there are reasons to argue that only the $u = u' = 1$ case corresponds to physical D-branes. Firstly, for $u, u' \neq 1$ the open string partition function is built from *non-unitary* $N = 2$ representations. An alternative argument was given in [43], based on comparing higher values of u to expectations for the physics of coinciding single D0-branes.

- From the case of the D0 branes we draw an important lesson, which will remain valid for the D1 and D2 branes. Although the open string partition function is built out of $N = 2$ characters, the product of the one point functions in the closed string channel is the same as that of the *bosonic* 2D black hole studied in [43] at level $k+2$. Using identities developed in [57] (see also [59]) to relate sums of characters of $N = 2$ representations to characters of a bosonic $SL(2,\mathbb{R})/U(1)$ model, one can expand the partition function (4.4) of a string stretching between a u-brane and a basic brane as

$$Z_{u,1}^{NS}(\tau,\nu) = \frac{1}{\eta(\tau)} \sum_{n,r\in\mathbb{Z}} z^{\frac{(j+r)}{k/2}+n} q^{\frac{k/2}{k+2}(\frac{(j+r)}{k/2}+n)^2} \left[\lambda_{j,r-n}(\tau) - \lambda_{-j+1,r-n-u}(\tau)\right] \quad (4.28)$$

where $j = -\frac{(u-1)}{2}$ and

$$\lambda_{j,r}(\tau) = \eta(\tau)^{-2} q^{-\frac{(j-\frac{1}{2})^2}{k}+\frac{(j+r)^2}{k+2}} \sum_{s=0}^{\infty} (-1)^s q^{\frac{1}{2}s(s+2r+1)} \quad (4.29)$$

are the characters of the bosonic coset $SL(2,\mathbb{R})/U(1)$ descending from bosonic $SL(2,\mathbb{R})$ primaries with $J_0^3 = j + r$. On the other hand, the open string partition function of a bosonic open string stretching between similar D-branes in the bosonic cigar background is given by [43]

$$Z_{u,1}^{bosonic}(\tau,\nu) = \sum_{r\in\mathbb{Z}} \left[\lambda_{j,r}(\tau) - \lambda_{-j+1,r}(\tau)\right] . \quad (4.30)$$

Notice that the partition functions (4.28) and (4.30) differ by characters of a $U(1)$ boson. This is the $U(1)$ R-current of $N = 2$, who is responsible for the extension of the bosonic $SL(2,\mathbb{R})/U(1)$ algebra into $N = 2$ [23]. It is a free boson, and it is coupled in a way that is of mild consequence to the modular matrix and to the one-point functions.

5. D0 branes in a $\mathbb{Z}_p$ orbifold

Before proceeding to the D1 branes, we will discuss a natural and interesting generalization of the D0 branes considered in the previous section.

In the annulus partition function (4.4) we summed over the whole infinite spectral flow orbit, and this was essential in order to obtain a discrete spectrum of $U(1)$ charges in the closed string channel. But a consistent result is also obtained if the sum is taken with

jumps of p units of spectral flow. As we will see, this correspond to D0 branes living in a $\mathbb{Z}_p$ orbifold of the original background, and in this framework we will be able to connect with previous works on boundary $N = 2$ Liouville theory [10, 11, 12].

The orbifold acts freely as a shift of $2\pi/p$ on the angular direction of the cigar. As for a compact free boson, we expect that the orbifold theory is equivalent to the original one with the radius divided by p. This implies a spectrum of charges given by

$$J_0^3 + \bar{J}_0^3 = m + \bar{m} = \frac{kw}{p} \ , \quad J_0^3 - \bar{J}_0^3 = m - \bar{m} = pn. \tag{5.1}$$

Moreover, since the (infinite) volume of the target space decreases now by a factor $1/p$, we expect that the one point functions should be renormalized by $\sqrt{1/p}$. For k integer one can mod out the theory by $\mathbb{Z}_k$ (see below), and we get the spectrum of single cover of the vector coset $SL(2,\mathbb{R})/U(1)$ [25], which has the target space geometry of the trumpet, at first order in $1/k$.

By applying the same method of descent from H_3^+ we can compute the one-point function for this orbifold. For simplicity, we will study here the NS/NS case, with $u = u' = 1$, and we omit the NS/u/u' labels.. The other sectors are similar.

We obtain the following one-point function for the D0-branes :

$$\Psi_{p,a}(j,w) \ = \ \frac{1}{\sqrt{pk}} e^{-2\pi i a \frac{w}{p}} \nu^{\frac{1}{2}-j} \frac{\Gamma(j+\frac{kw}{2p})\Gamma(j-\frac{kw}{2p})}{\Gamma(2j-1)\Gamma(1-\frac{1-2j}{k})} \tag{5.2}$$

where $a \in \mathbb{Z}_p$ is an additional quantum number of these D-branes and the phase $e^{-2\pi i a \frac{w}{p}}$ is fixed by the Cardy condition.

Let us start with the open string partition function for an open string stretching between two such D0-branes, for which we consider

$$\begin{aligned} Z_{p;a,a'}(\tau,\nu) \ &= \sum_{r \in \mathbb{Z}p+a-a'} ch_f(1,r;\tau,\nu) \\ &= \frac{\vartheta_3(\tau,\nu)}{\eta(\tau)^3} \left[\sum_{s \in \mathbb{Z}p+a-a'+\frac{1}{2}} \frac{q^{\frac{s^2-s}{k}} y^{\frac{2s-1}{k}}}{1+yq^s} - \sum_{s \in \mathbb{Z}p-1+a-a'+\frac{1}{2}} \frac{q^{\frac{s^2+s}{k}} y^{\frac{2s+1}{k}}}{1+yq^s} \right] . \end{aligned} \tag{5.3}$$

We have summed over all the representations of the spectral flow with p-jumps, with the starting point $r = a - a'$ coming from the additional parameters of these D0-branes.

In order to check the Cardy condition, we will perform a modular transformation of $Z_{p;a,a'}$ to the closed string channel. The computation is very similar to that of the previous section, so we will indicate only the major steps. We start with

$$\begin{aligned} e^{-i\pi \frac{c}{3}\frac{\nu^2}{\tau}} \quad Z_{p;a,a'}(-\frac{1}{\tau},\frac{\nu}{\tau}) = \frac{\vartheta_3(\tau,\nu)}{\eta(\tau)^3} \ \times \\ \Bigg\{ \frac{1}{2i\pi} \left[\int_{\mathcal{C}_{-\epsilon}} + \int_{\mathcal{C}_{+\epsilon}} \right] dZ \ \frac{(-i\pi)}{p} \frac{e^{\pi Z + 2\pi \frac{Z}{k} + \frac{2i\pi\tau}{k} Z^2}}{\cosh(\pi Z)} \frac{e^{-i\pi(\frac{\nu - i\tau Z - 1/2 - (a-a')}{p} + \frac{1}{2})}}{2\cos\pi(\frac{\nu - i\tau Z - 1/2 - (a-a')}{p} + \frac{1}{2})} \\ - \ \frac{1}{2i\pi} \left[\int_{\mathcal{C}_{-\epsilon}} + \int_{\mathcal{C}_{+\epsilon}} \right] dZ \ \frac{(-i\pi)}{p} \frac{e^{\pi Z - 2\pi \frac{Z}{k} + \frac{2i\pi\tau}{k} Z^2}}{\cosh(\pi Z)} \frac{e^{-i\pi(\frac{\nu - i\tau Z + 1/2 - (a-a')}{p} + \frac{1}{2})}}{2\cos\pi(\frac{\nu - i\tau Z + 1/2 - (a-a')}{p} + \frac{1}{2})} \Bigg\} . \end{aligned} \tag{5.4}$$

The contour of integration is the same as in the previous section, but now the last factor of the integrand has poles at values $\nu - i\tau Z = p\mathbb{Z} + a - a' + \frac{1}{2}$ in the first term, and at $\nu - i\tau Z = p\mathbb{Z} + a - a' - \frac{1}{2}$ for the second. Expanding the last factor of (5.4) as in (4.9)-(4.10), and taking $\epsilon \to 0$ in the contours $\mathcal{C}_{\pm\epsilon}$, yields,

$$
\begin{aligned}
& e^{-i\pi\frac{c}{3}\frac{\nu^2}{\tau}} Z_{p;a,a'}(-\frac{1}{\tau}, \frac{\nu}{\tau}) = \\
& \qquad \frac{\vartheta_3(\tau,\nu)}{\eta(\tau)^3} \sum_{w\in\mathbb{Z}} \int_{\mathcal{C}} dZ \; e^{-2i\pi(a-a')\frac{w}{p}} \frac{\sinh(\pi(\frac{2Z}{k} - \frac{iw}{p}))}{p\cosh(\pi Z)} e^{\pi Z} \; y^{\frac{w}{p}} q^{-i\frac{wZ}{p} + \frac{Z^2}{k}} . \qquad (5.5)
\end{aligned}
$$

In order to express (5.5) as a sum over characters we shift the contour in each term by an amount of $\frac{iwk}{2p} - i\nu_2$. As in the previous section, we pick poles for $Z = is$, with $s \in \mathbb{Z} + \frac{1}{2}$, corresponding to the discrete representations. Defining $w_s \in \mathbb{Z}$ as

$$
w_s > \frac{2ps}{k} > w_s - 1 \, , \qquad (5.6)
$$

the poles with $s > \nu_2$ ($s < \nu_2$) are picked with positive (negative) sign, by all the values $w \geq w_s$ ($w < w_s$) in (5.5). The sum of all the poles is

$$
\begin{aligned}
& -2\frac{\vartheta_3(\tau,\nu)}{\eta(\tau)^3} \left[\sum_{s>\nu_2} \sum_{w=w_s}^{\infty} - \sum_{s<\nu_2} \sum_{-\infty}^{w_s-1} \right] e^{-2\pi i(a-a')\frac{w}{p}} \sin\left(2\pi\frac{s}{k} - \frac{\pi w}{p}\right) \; y^{\frac{w}{p}} q^{\frac{sw}{p} - \frac{s^2}{k}} \qquad (5.7) \\
= & -2\frac{\vartheta_3(\tau,\nu)}{\eta(\tau)^3} \sum_{s\in\mathbb{Z}+\frac{1}{2}} \sum_{n\in\mathbb{Z}_p} e^{-2\pi i(a-a')\frac{w_s+n}{p}} \sin\left(2\pi\frac{s}{k} - \pi\frac{(w_s+n)}{p}\right) \; \frac{y^{\frac{w_s+n}{p}} q^{\frac{s(w_s+n)}{p} - \frac{s^2}{k}}}{1+yq^s} .
\end{aligned}
$$

Collecting the continuous and discrete contributions, we get finally

$$
\begin{aligned}
& e^{-i\pi\frac{c}{3}\frac{\nu^2}{\tau}} Z_{p;a,a'}(-\frac{1}{\tau}, \frac{\nu}{\tau}) = \qquad (5.8) \\
& \quad \sum_{w\in\mathbb{Z}} \int_0^{+\infty} dP \; \Psi_{p,a'}\left(\frac{1}{2} - iP, -w\right) \Psi_{p,a}\left(\frac{1}{2} + iP, w\right) \; ch_c^{NS}(P, \frac{kw}{2p}; \tau, \nu) \\
& \quad +2\pi \sum_{r\in\mathbb{Z}} \sum_{n\in\mathbb{Z}_p} Res\left[\Psi_{p,a'}(-j_{r,n}+1, -w_s-n)\Psi_{p,a}(j_{r,n}, w_s+n)\right] \; ch_d^{NS}(j_{r,n}, r; \tau, \nu) \, ,
\end{aligned}
$$

where as usual the residue is computed when considering the bracketed expression as a function of j. The spins $j_{r,n}$ of the discrete representations are defined through

$$
j_{r,n} + r = \frac{k(w_s+n)}{2p} \qquad (5.9)
$$

where $s = r + \frac{1}{2}$, and $j_{r,n}$ satisfies the unitary bound (2.7). Moreover, from (5.6) it follows that the width $k/2$ of the unitary bound (2.7) is sliced into p intervals, and for each $n \in \mathbb{Z}_p$ we have

$$
\frac{1}{2} + \frac{kn}{2p} < j_{r,n} < \frac{k(n+1)}{2p} + \frac{1}{2} \, . \qquad (5.10)
$$

Finally, note that the choice of a in the open string channel, which is a statement about the *spectrum*, becomes a *phase* in the closed string channel.

The $\mathbb{Z}_k$ and $\mathbb{Z}_\infty$ cases

In this context, we can now construe previous studies of D-branes in $N = 2$ Liouville theory as particular cases of the orbifold discussed here.

In [10] D-branes were built in a $N = 2$ model with rational central charge $c = 3 + \frac{6K}{N}$, with K, N positive integers, and the D0-brane open string partition function was taken with jumps of N spectral flow units (for $u = u' = 1$). This corresponds to level $k = \frac{N}{K}$ and $p = N$ in our case. In particular, for $K = 1$, the asymptotic radius shrinks from $R = \sqrt{k\alpha'}$ to $\frac{R}{k} = \frac{\alpha'}{R}$, which is the T-dual radius[18]. Indeed, notice that for $p = k$, the spectrum of pure winding modes in (5.1) becomes pure momentum in the T-dual picture. Note that we strengthen some of the results in [10], since our construction is explicitly based on branes which have been checked to satisfy factorization constraints in the parent theory [49]. Moreover, our results our based on a systematic analysis of a solution to the branes preserving the full chiral algebra in the parent theory, and do not depend on the central charge being rational. Apart from these bonuses, we also clarified the connection to the semi-classical geometrical (mirror) cigar-picture.

Another interesting case to consider is the limit $p \to \infty$. This corresponds to the cigar radius shrinking to zero, so that the spectrum (5.1) of m for pure winding states becomes continuous, thus yielding a continuum of R-charges in the closed string channel. As shown in [25] it corresponds to the vector gauging of the *universal cover* of $SL(2,\mathbb{R})$.

At the level of the Cardy computation, naively taking the limit $p \to \infty$ in (5.8) gives zero in the closed string channel, since $\Psi_{p,a}(j, w)$ goes to zero, see (5.2). The correct way of proceeding is to notice that equation (5.5) becomes a Riemann sum, which leads to an integral over $t = \frac{w}{p} \in \mathbb{R}$. We can moreover express $t = x + g,\ g \in \mathbb{Z},\ x \in [0, 1)$. The sum over w in (5.5) becomes

$$\frac{1}{p}\sum_{w} \longrightarrow \int_{-\infty}^{+\infty} dt \longrightarrow \int_0^1 dx \sum_{g\in\mathbb{Z}} . \tag{5.11}$$

One can show that the resulting expression for the closed string channel is equivalent to starting with

$$Z_{1,1}(x;\tau,\nu) = \sum_{r\in\mathbb{Z}} e^{2\pi i x r} ch_f^{NS}(1, r; \tau, \nu)\,, \tag{5.12}$$

and computing

$$\int_0^1 dx\ e^{-2\pi i x d} e^{-i\pi\frac{c}{3}\frac{\nu^2}{\tau}} Z_{1,1}(x, -\frac{1}{\tau}, \frac{\nu}{\tau})\,, \tag{5.13}$$

where $d \in \mathbb{Z}$ depends on the precise way the limit is taken in (5.11). The computation of (5.13) itself can be performed with an identity proved in [56, 57], to which we refer for details. The resulting expression can be found in [12].

[18]In the case of rational k there will be additional discrete terms in the closed string channel partition function, coming from poles falling exactly on the displaced contour of section 4. But taking them into account does not change the orbifold picture.

6. D1 branes

In this section, we check the relative Cardy condition for D1-branes in the $N = 2$ Liouville/supersymmetric coset conformal field theory. In this and the next sections, we will work in the NS sector. The other sectors are obtained straightforwardly. Following the general logic, we assume that the one-point functions for closed string primaries in the presence of a D1-brane labeled by the parameters (r, θ_0) is given by (see Appendix B):

$$\langle \Phi^j_{nw}(z, \bar{z}) \rangle^{D1}_{r,\theta_0} = \delta_{w,0} \, \frac{\Psi_{r,\theta_0}(j, n)}{|z - \bar{z}|^{\Delta_{j,n}}} \ , \tag{6.1}$$

with

$$\Psi_{r,\theta_0}(j, n) = \mathcal{N}_1 e^{in\theta_0} \left\{ e^{-r(-2j+1)} + (-1)^n e^{r(-2j+1)} \right\} \nu^{\frac{1}{2}-j} \frac{\Gamma(-2j+1)\Gamma(1+\frac{1-2j}{k})}{\Gamma(-j+1+\frac{n}{2})\Gamma(-j+1-\frac{n}{2})} \tag{6.2}$$

We put a normalization factor $\mathcal{N}_1$ up front that will be fixed during the Cardy computation.

Note that the only poles in the one-point function originate from infinite volume divergences (i.e. they are bulk poles). Thus the only contributions to the closed string channel amplitude between two D1-branes originate from continuous representations (unlike the D0- and D2-brane computations where poles associated to discrete representations require special attention). The closed string channel amplitude will only contain contributions proportional to the continuous character:

$$ch^{NS}_c(P, \frac{n}{2}; -\frac{1}{\tau}, \frac{\nu}{\tau}) = \tilde{q}^{\frac{P^2}{k} + \frac{n^2}{4k}} \tilde{y}^{\frac{n}{k}} \frac{\vartheta_3(-\frac{1}{\tau}, \frac{\nu}{\tau})}{\eta^3(-\frac{1}{\tau})}. \tag{6.3}$$

We compute the closed string channel amplitude between two branes with identical boundary conditions both labeled by (r, θ_0), for simplicity. The computation can then easily be generalized to include the case of differing boundary conditions, following [60]. For our case, we find the partition function:

$$\begin{aligned} Z^{D1}_{r,\theta_0} &= \int_0^\infty dP \sum_{n \in \mathbb{Z}} \Psi_{r,\theta_0}(P, n) \Psi_{r,\theta_0}(-P, -n) \; ch^{NS}_c \left(P, \frac{n}{2}; -\frac{1}{\tau}, \frac{\nu}{\tau} \right) \\ &= \frac{4\mathcal{N}_1^2}{k} \int_0^\infty dP \; \frac{1}{\sinh 2\pi P \sinh \frac{2\pi P}{k}} \\ &\quad \left(\sum_{n \in 2Z} \cos^2 2rP \cosh^2 \pi P \; ch^{NS}_c(P, \frac{n}{2}) + \sum_{n \in 2Z+1} \sin^2 2rP \sinh^2 \pi P \; ch^{NS}_c(P, \frac{n}{2}) \right). \end{aligned} \tag{6.4}$$

We modular transform the characters to obtain the annulus amplitude suitable for interpretation in the open string channel:

$$\begin{aligned} e^{-\frac{ci\pi\nu^2}{3\tau}} Z^{D1}_{r,\theta_0} &= \frac{8\mathcal{N}_1^2}{k} \int_0^\infty dP \int_0^\infty dP' \sum_{w \in \frac{\mathbb{Z}}{2}} \cos \frac{4\pi P P'}{k} \\ &\quad \times \frac{\cos^2 2rP \cosh^2 \pi P + (-1)^{2w} \sin^2 2rP \sinh^2 \pi P}{\sinh 2\pi P \sinh \frac{2\pi P}{k}} \; ch^{NS}_c(P', kw; \tau, \nu) \end{aligned} \tag{6.5}$$

Since the transformation properties of continuous $N = 2$ characters are analogous to the transformation properties of purely bosonic coset characters, we can discuss the results similarly as for the bosonic coset [43]. To make the discussion more explicit, and in particular to match on to the boundary reflection amplitude, we follow [49, 43] to check the relative Cardy condition. We compare the regularized density of states obtained in the open string channel with the one we expect from a reflection amplitude which is the natural $N = 2$ generalization of the reflection amplitude in [49, 60, 43] (which satisfies factorization). To this end, we subtract a reference amplitude labeled by r^*, and use trigonometric identities and the change of variables $t = 2\pi P$ to obtain:

$$e^{-\frac{ci\pi\nu^2}{3\tau}}(Z^{D1}_{r,\theta_0} - Z^{D1}_{r^*,\theta_0}) = \frac{8\mathcal{N}_1^2}{k}\int_0^\infty dP' \sum_{w\in\frac{\mathbb{Z}}{2}} ch_c^{NS}(P', kw) \;\times \tag{6.6}$$

$$\frac{\partial}{\partial P'}\int_0^\infty \frac{dt}{t}\frac{k}{16\pi}\frac{\cosh^2\frac{t}{2} - (-1)^{2w}\sinh^2\frac{t}{2}}{\sinh t \sinh\frac{t}{k}}\left\{\sin\frac{2t}{k}(P' + \frac{rk}{\pi}) + \sin\frac{2t}{k}(P' - \frac{rk}{\pi}) - (r \to r^*)\right\}.$$

To link the relative density of continuous states in the open string channel to the boundary reflection amplitude it is convenient to define the special functions:

$$\begin{aligned}
\log S_k^{(0)}(x) &= i\int_0^\infty \frac{dt}{t}\left(\frac{\sin\frac{2tx}{k}}{2\sinh\frac{t}{k}\sinh t} - \frac{x}{t}\right), \\
\log S_k^{(1)}(x) &= i\int_0^\infty \frac{dt}{t}\left(\frac{\cosh t\sin\frac{2tx}{k}}{2\sinh\frac{t}{k}\sinh t} - \frac{x}{t}\right).
\end{aligned} \tag{6.7}$$

We can then write our ansatz for the boundary reflection amplitudes as:

$$\begin{aligned}
R(P, w\in\mathbb{Z}|r) &= \nu_k^{iP}\frac{\Gamma_k^2(-\frac{1}{2} - iP + k)\Gamma_k(2iP + k)S_k^{(0)}(P + \frac{rk}{\pi})}{\Gamma_k^2(\frac{1}{2} + iP + k)\Gamma_k(-2iP + k)S_k^{(0)}(-P + \frac{rk}{\pi})} \\
\tilde{R}(P, w\in\mathbb{Z}+\frac{1}{2}|r) &= \nu_k^{iP}\frac{\Gamma_k^2(-\frac{1}{2} - iP + k)\Gamma_k(2iP + k)S_k^{(1)}(P + \frac{rk}{\pi})}{\Gamma_k^2(\frac{1}{2} + iP + k)\Gamma_k(-2iP + k)S_k^{(1)}(-P + \frac{rk}{\pi})}.
\end{aligned} \tag{6.8}$$

For the definition of the generalized gamma-functions Γ_k, we refer to e.g. [43] – they immediately drop out of the computation of the relative partition function. Using the reflection amplitudes (and the fact that they are parity odd) we can show that the relative Cardy condition holds:

$$\begin{aligned}
e^{-\frac{ci\pi\nu^2}{3\tau}}(Z^{D1}_{r,\theta_0} - Z^{D1}_{r^*,\theta_0}) &= \frac{\mathcal{N}_1^2}{\pi i}\int_0^\infty dP'\left\{\sum_{w\in\mathbb{Z}}\left(\frac{\partial}{\partial P'}\log\frac{R(P', w|r)}{R(P', w|r^*)}\right)ch_c^{NS}(P', kw)\right. \\
&\left.+\sum_{w\in\mathbb{Z}+\frac{1}{2}}\left(\frac{\partial}{\partial P'}\log\frac{\tilde{R}(P', w|r)}{\tilde{R}(P', w|r^*)}\right)ch_c^{NS}(P', kw)\right\}.
\end{aligned} \tag{6.9}$$

To obtain agreement with the density of states expected on the basis of the boundary reflection amplitude, we can fix $\mathcal{N}_1^2 = 1/2$. Notice that in the open string channel, a pure

winding state has $J^3_{0,open} = 2J^3_{0,closed} = kw$. So we get in (6.9) contributions from open strings winding both integer and half-integer times around the cigar. This is consistent with the semiclassical geometry of the D1 branes in the cigar [43].

We have thus verified that the relative Cardy condition is satisfied by our $D1$-branes. In summary, in this section we have argued that the relative Cardy condition holds, given the one-point functions for the D1-branes we started out with, and the extension of the boundary reflection amplitudes of [49, 60] to $N = 2$ theories. The computation follows the lines of [43] due to the close connection between (continuous) $N = 2$ characters and those of the bosonic coset, and their modular properties (see comment at the end of Section 4). Note that the D1-branes couple to continuous bulk states with zero winding only.

7. D2-branes

In this section, we analyze the Cardy condition for the D2-branes which we can construct from the H_2 branes in Euclidean AdS_3. They correspond to type B branes w.r.t. the N=2 superconformal algebra. We find some puzzling features when trying to perform the Cardy check. Following [43] and the general logic outlined before, we propose the following one-point function for the D2-branes parameterized by σ, in the NS sector :

$$\langle \Phi^j_{nw}(z,\bar z)\rangle^{D2}_\sigma = \delta_{n,0}\, \frac{\Psi_\sigma(j,w)}{|z-\bar z|^{\Delta_{j,w}}}\ , \quad \text{with :}$$
$$\Psi_\sigma(j,w) = \mathcal{N}_2\, \nu^{\frac12 - j}\, \frac{\Gamma(j+\frac{kw}{2})\Gamma(j-\frac{kw}{2})}{\Gamma(2j-1)\Gamma(1-\frac{1-2j}{k})}\, \frac{e^{i\sigma(1-2j)}\sin\pi(j-\frac{kw}{2}) + e^{-i\sigma(1-2j)}\sin\pi(j+\frac{kw}{2})}{\sin\pi(1-2j)\sin\pi\frac{1-2j}{k}} \tag{7.1}$$

This one point function has poles corresponding to the discrete representations, and therefore will couple both to localized and extended states. This is expected on general grounds since these D2-branes carry D0-brane charge.

The annulus partition function in the closed string channel, for general Casimir labeled by j, in the NS sector, is:

$$Z^{D2}_{\sigma\sigma'}(-1/\tau,\nu/\tau) = -k\mathcal{N}_2^2 \int dj \sum_{w\in\mathbb{Z}} \frac{ch^{NS}\left(j,\frac{kw}{2};-1/\tau,\nu/\tau\right)}{\sin\pi(1-2j)\sin\pi\frac{(1-2j)}{k}} \tag{7.2a}$$
$$\Big\{ 2\cos(\sigma+\sigma')(1-2j) - 2\cos(\sigma-\sigma')(1-2j)\cos 2\pi j \tag{7.2b}$$
$$+ \frac{2\cos(\sigma-\sigma')(1-2j)\sin^2 2\pi j}{\cos\pi kw - \cos 2\pi j} - \frac{2i\sin(\sigma-\sigma')(1-2j)\sin 2\pi j \sin\pi kw}{\cos\pi kw - \cos 2\pi j}\Big\} \tag{7.2c}$$

We can read from this expression that the different terms will contribute in a very different fashion.

7.1 D1-like contribution

The two terms of the second line (7.2b) will give a contribution similar to the D1-branes (which is an expected contribution, on the basis of the fact that the D2-branes also stretch

along the radial direction), with imaginary parameter though. Explicitely, we have in the closed channel :

$$Z^{D2,(b)}_{\sigma\sigma'}(-1/\tau,\nu/\tau) = 2k\mathcal{N}_2^2 \int_0^\infty dP \sum_{w\in\mathbb{Z}} \frac{ch_c^{NS}\left(P,\frac{kw}{2};-1/\tau,\nu/\tau\right)}{\sinh 2\pi P \sinh 2\pi P/k} \left[\cosh 2P(\sigma+\sigma') + \cosh 2P(\sigma-\sigma')\cosh 2\pi P\right] . \quad (7.3)$$

As for the D1-branes, we consider the *relative partition function* w.r.t. the annulus amplitude for reference branes of parameters (σ_0,σ_0'). Going through the same steps as in the previous sections, we obtain the open string channel amplitude :

$$Z^{D2,(b)}_{\sigma\sigma'}(\tau,\nu) = 4k\mathcal{N}_2^2 \int dP' \frac{\partial}{2i\pi\,\partial P'} \log\left\{\frac{R(P'|i\frac{\sigma+\sigma'}{2})\,\tilde{R}(P'|i\frac{\sigma-\sigma'}{2})}{R(P'|i\frac{\sigma_0+\sigma_0'}{2})\,\tilde{R}(P'|i\frac{\sigma_0-\sigma_0'}{2})}\right\} \sum_{n\in\mathbb{Z}} ch_c^{NS}(P',n;\tau,\nu) \quad (7.4)$$

in terms of reflections amplitudes similar as before, see (6.8), but with imaginary parameters $i(\sigma\pm\sigma')/2$. We fix the normalization constant to $\mathcal{N}_2^2 = \frac{1}{4k}$.

7.2 D0-like contribution

Now we concentrate on the last two terms (7.2c) of the annulus amplitude. As we will see this will give a contribution similar to those of D0-branes, hence with both discrete and continuous contributions. Let's first concentrate on the latter. Since the last term is odd in w, it will cancel from the amplitude[19] and we are left with :

$$Z^{D2,(c)}_{\sigma\sigma',\,cont}(-1/\tau,\nu/\tau) = -\frac{1}{2}\int_0^\infty dP \sum_{w\in\mathbb{Z}} \frac{\cosh 2P(\sigma-\sigma')\sinh^2 2\pi P\ ch_c^{NS}\left(P,\frac{kw}{2};-1/\tau,\nu/\tau\right)}{(\cosh 2\pi P+\cos\pi kw)\sinh 2\pi P\sinh 2\pi P/k} \quad (7.5)$$

Assuming that $\sigma-\sigma' = 2\pi m/k$, $m\in\mathbb{Z}$, we recognize a D0 amplitude for two branes of same parameter m :

$$Z^{D2,(c)}_{\sigma\sigma',\,cont}\left(-\frac{1}{\tau},\frac{\nu}{\tau}\right) = -\frac{1}{2}\int_0^\infty dP \sum_{w\in\mathbb{Z}} \frac{\left(2\sinh^2(2\pi Pm/k)+1\right)\sinh 2\pi P}{(\cosh 2\pi P+\cos\pi kw)\sinh 2\pi P/k} \times \quad ch_c^{NS}\left(P,\frac{kw}{2};-1/\tau,\nu/\tau\right) , \quad (7.6)$$

up to the constant term in the bracketed expression, that will drop from the relative partition function. The normalization $-1/2$ of this expression has to be compared with the normalization $(-)^{w(m-m)} = 1$ of the D0 computation (4.16).

Discrete representations

We can also make the identification with a D0-like contribution as follows. First we consider the more straightforward case $w>0$. Then we pick the poles of the discrete representations

[19]Strictly speaking, this holds only when $\nu=0$. The same is true in the discrete sector below.

in the domain : $j \in D = \left(\frac{kw}{2} - \mathbb{N}\right) \cap \left]\frac{1}{2}; \frac{k+1}{2}\right[$. For each pole we will have a contribution of 2π times the residue :

$$-\frac{1}{2}\sum_{j\in D}\sum_{w>0}(-)^{j-\frac{kw}{2}}\frac{\cos 2\pi m\frac{2j-1}{k}\ \sin\pi(2j-1) - i\sin 2\pi m\frac{2j-1}{k}\ \sin\pi kw}{\sin\pi(j+\frac{kw}{2})\sin\pi\frac{2j-1}{k}} ch_d^{NS}(j, -kw/2-j) \tag{7.7}$$

We write $j = \frac{kw}{2} - r$, with $r \in \mathbb{N}$. Then for every r, there is only one value of w, that we'll call w_r such that j is in the correct range. Explicitely w_r is given by: $w_r = \lfloor \frac{2r+1}{k} \rfloor + 1$. We call also j_r the value of j that has been picked. With this procedure we get :

$$-\frac{1}{2}\sum_{r\in\mathbb{N}}(-)^{w_r}\frac{\cos 2\pi m\frac{2r+1}{k} - i\sin 2\pi m\frac{2r+1}{k}}{\sin\pi\frac{2r+1}{k}} ch_d^{NS}(j_r, r) \tag{7.8}$$

Let us now consider the case $w \leqslant 0$. In this case we have to use the isomorphism of representations : $\mathcal{D}_j^{+,w} \simeq \mathcal{D}_{j'}^{-,w-1}$, with $j' = \frac{k+2}{2} - j$. We have then $j' = -\frac{k}{2}(w-1) - r'$, $r' \in \mathbb{N}$, and the value of w is fixed to : $w_{r'} = -\lfloor \frac{2r'+1}{k} \rfloor$. This leads to the following contribution to the annulus amplitude:

$$-\frac{1}{2}\sum_{r'\in\mathbb{N}}(-)^{w_r'-1}\frac{\cos 2\pi m\frac{2r'+1}{k} + i\sin 2\pi m\frac{2r'+1}{k}}{\sin\pi\frac{2r'+1}{k}} ch_d^{NS}(j'_{r'}, -r') \tag{7.9}$$

Then it is possible to add the two contributions, which cancels the imaginary part, and leaves us with :

$$Z^{D2,(c)}_{\sigma\sigma',\,disc}\left(-\frac{1}{\tau},\frac{\nu}{\tau}\right) = -\frac{1}{2}\sum_{r\in\mathbb{Z}}(-)^{\lfloor\frac{2r+1}{k}\rfloor}\frac{2\sin^2 2\pi m\frac{r+1/2}{k} - 1}{\sin 2\pi\frac{r+1/2}{k}} ch_d^{NS}\left(j_r, r; -\frac{1}{\tau}, \frac{\nu}{\tau}\right). \tag{7.10}$$

This is again $-1/2$ of the expression of the amplitude for two D0's of parameter m, eq. (4.22), up to the irrelevant constant term.

Some comments on D2-branes physics

The following comments are in order:

- in the computation we assumed that the difference of the parameters of the D2 in the annulus amplitude is quantized: $\sigma' - \sigma = 2\pi m/k$, $m \in \mathbb{Z}$. This relative quantization condition is discussed in [43] for the bosonic coset. Indeed the difference of the D2-branes parameters is the net induced D0-charge, hence it should be quantized. To be more precise it is believed that a D2-brane with parameter σ' is a bound state of a brane of parameter σ with m D0-branes (for the $\sigma' > \sigma$ case , otherwise one has to reverse the picture)

- as a corollary, the one-point functions for the D2-branes have poles *both* of the localized type and of the bulk type

- the computation of the annulus amplitude gives a continuous spectrum of open strings attached to the D2-branes, with a sensible density of states, and also a contribution

similar to the D0 – see the previous remark about the induced D0 charge – but with a normalization $(-1/2)$ which complicates the task of making sense of the open string spectrum, and blurs the physical picture of bound states. The only open string spectrum leading to a good physical picture would seem to correspond to two D2-branes with the same parameter σ.

Clearly further study of the physics of D2-branes is needed to clarify their interpretation.

8. Conclusions

We constructed D-branes in $N = 2$ Liouville theory and the $SL(2,\mathbb{R})/U(1)$ super-coset conformal field theory, and checked the (relative) Cardy condition as well as consistency with the proposed boundary reflection amplitudes, which were proven earlier to satisfy the factorization constraints. The one-point functions that we constructed remarkably decouple the poles in bulk amplitudes (notably the reflection amplitude) into poles associated to infinite volume and the ones associated to normalizable discrete states. It would be interesting to investigate whether the boundary decoupling phenomenon aids in understanding more aspects of the conjectured holographic duality in the (doubly scaled) little string theory.

We have shown (for the particular but extendable) case of the D0-branes how to generalize the results to other sectors of the supersymmetric Hilbert space, and to orbifolds of the cigar conformal field theory.

We have pointed out throughout this work some similarities of the one-point functions for $N = 2$ Liouville with those of the $N = 0, 1$ cases, like the decoupling of the bulk poles. It would be interesting to see also whether the interesting relations uncovered in [61, 62] between the boundary states associated to localized and extended branes have also any manifestation in the $N = 2$ theory.

In appendices, we argued for the general use of the $SL(2,\mathbb{R})$ symmetry that can be obtained by enhancing $N = 2$ theories with an orthogonal free scalar, and we suggested that this tool provides some technical support for an attempt to interpret the variables $(x, \bar{x})$ – parameterizing the $SL(2,\mathbb{R})$ quantum numbers in position space on the boundary of AdS_3 – as new worldsheet variables. We showed in great detail the fact that super-coset characters agree with $N = 2$ characters, and the $N = 2$ spectral flow has a natural interpretation in terms of $SL(2,\mathbb{R})$ quantum numbers.

Note that we have constructed D-branes for generic values of the level k. It is known that the conformal field theory correlators depend strongly on whether k is rational or irrational (for instance via the shift equations, or the structure of the poles in the bulk correlators). It seems important to further clarify the relation between the construction of branes at rational and irrational values of k (for instance by further comparing the techniques developed in [57] to the results of [43]), as well as the dependence of bulk correlators on this most intriguing distinguishing feature.

The branes we constructed form an integral part of the construction of D-branes in non-compact non-trivially curved supersymmetric string theory backgrounds. One particular

application amongst these is the construction of D1-branes and D3-branes in the Little String Theory background in the double scaling limit (see also e.g. [63, 64, 65]). Indeed, when we consider the doubly scaled limit for NS5-branes which are evenly distributed over a topologically trivial circle, we obtain a closed string background which is supposed to represent a holographic dual of the Higgs phase of the Little String Theory living on the NS5-branes [5], whose partition function has been studied in [26, 8]. The role of the W-bosons in the LST is played by the D1-branes stretching between the NS5-branes. One can construct at least the one-point functions corresponding to these D-branes by combining the D0-brane in the $SL(2,\mathbb{R})/U(1)$ theory and a D1-brane in the supersymmetric $SU(2)/U(1)$ coset theory, properly taking into account the discrete orbifold operation. The construction of these one-point functions is now straightforward. We will return to some of these issues in the near future [66].

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A. Computing N=2, c>3 characters

In this appendix we will compute the $N = 2, c > 3$ characters appearing in the spectrum of the D-branes studied in the paper. We will obtain them as characters of the supersymmetric coset $SL(2,\mathbb{R})/U(1)$, through a rather standard procedure. The same characters can be computed by subtracting the null-vectors modules from the free action of the $N = 2$ generators, as has been done in [67, 68, 69]. The $N = 2$ representations that we consider are both unitary and non-unitary. The characters of the former have been computed in those papers and coincide with our results, while for non-unitary representations we present the characters for the first time. The fact that the coset yields *irreducible* $N = 2$ characters can be traced back to the fact that the $SL(2,\mathbb{R})$ characters we start with correspond to irreducible representations of the $SL(2,\mathbb{R})$ algebra.

Notice that we have here a non-minimal version of a similar situation for ($N = 1, 2$ supersymmetric) unitary minimal models, where the (supersymmetric) Virasoro characters coincide with the characters of certain cosets involving $SU(2)$ factors which realize the minimal models [70].

Moreover, this coincidence of the two ways of computing the characters, which was independently noticed in [26] and [8], is central to the fact that the D-branes that we build

in this paper belong to the class of objects, such as the correlation functions [4, 35, 32] or the modular-invariant partition function [26, 8], for which there is no distinction as to whether we are in the $N=2$ Liouville or in the supersymmetric cigar.

Let us review first how the $N=2$ algebra arises in the susy $SL(2,\mathbb{R})/U(1)$ coset [3]. The supersymmetric $SL(2,\mathbb{R})$ model at level k has currents J^a, ψ^b $(a,b=1,2,3)$, with OPEs

$$\begin{aligned}
J^a(z)J^b(w) &\sim \frac{g^{ab}k/2}{(z-w)^2} + \frac{f^{ab}{}_c\, J^c(w)}{z-w}\,, \\
J^a(z)\psi^b(w) &\sim \frac{f^{ab}{}_c\psi^c(w)}{z-w}\,, \\
\psi^a(z)\psi^b(w) &\sim \frac{g^{ab}}{z-w}
\end{aligned} \tag{A.1}$$

where $g^{ab} = \mathrm{diag}(+,+,-)$, $f^{123}=1$ and indices in the antisymmetric f^{abc} are raised and lowered with g^{ab}.

We first define

$$\begin{aligned}
j^a &= J^a - \hat{J}^a \\
\hat{J}^a &= -\frac{i}{2} f^a{}_{bc}\psi^b\psi^c
\end{aligned} \tag{A.2}$$

The currents j^a commute with the three fermions and generate a *bosonic* $SL(2,\mathbb{R})$ model at level $k+2$. The currents $\hat{J}^a, \psi^a$ form a supersymmetric $SL(2,\mathbb{R})$ model at level -2. The Hilbert space of the original supersymmetric $SL(2,\mathbb{R})_k$ theory is the direct product of the Hilbert space of the bosonic $SL(2,\mathbb{R})_{k+2}$ and that of the three free fermions.

We are interested in the coset obtained by gauging the $U(1)$ symmetry generated by J^3, ψ^3. This coset has an $N=2$ algebra generated by

$$\begin{aligned}
G^{\pm} &= \sqrt{\frac{2}{k}}\;\psi^{\pm} j^{\mp} \\
J^R &= \frac{2}{k} j^3 + \frac{k+2}{k}\;\hat{J}^3 = \frac{2}{k}J^3 + \psi^+\psi^- \\
T &= T_{SL(2,\mathbb{R})} - T_{U(1)}
\end{aligned} \tag{A.3}$$

where $\sqrt{2}\psi^{\pm} = \psi^1 \pm i\psi^2$ and

$$T_{U(1)} = -\frac{1}{k}J^3J^3 + \frac{1}{2}\psi^3\partial\psi^3\,. \tag{A.4}$$

The currents (A.3) commute with J^3, ψ^3 and satisfy the $N=2$ superconformal algebra

$$\begin{aligned}
J^R(z)J^R(w) &\sim \frac{c/3}{(z-w)^2}\,, \\
J^R(z)G^{\pm}(w) &\sim \pm\frac{G^{\pm}(w)}{(z-w)}\,, \\
G^+(z)G^-(w) &\sim \frac{2c/3}{(z-w)^3} + \frac{2J^R(w)}{(z-w)^2} + \frac{1}{(z-w)}\left(2T(w) + \partial J^R(w)\right)\,.
\end{aligned} \tag{A.5}$$

with the central charge

$$c = 3 + \frac{6}{k} \,. \tag{A.6}$$

A highest-weight representation of the $N = 2$ algebra with a given central charge c is determined by the conformal dimension h and the $U(1)$ charge Q of the highest weight state. By building the $N = 2$ representations through the $SL(2,\mathbb{R})/U(1)$ coset, we can parameterize h, Q by means of $SL(2,\mathbb{R})$ quantum numbers as follows from the zero modes of J^3, T in (A.3).

The states of the coset are all those states in the parent $SL(2,\mathbb{R})$ theory annihilated by the modes $J^3_{n>0}, \psi^3_{n>0}$, and a *primary* state of the coset is a coset state which is also a primary of the $N = 2$ algebra. Every primary state of the parent theory is clearly a primary of the coset. In addition, for the discrete and finite-dimensional representations of j^a, there are also *descendent* states of the parent theory which are *primaries* of the coset.

The Hilbert space of the parent theory can be decomposed into subspaces with definite J^3_0 eigenvalue m. Since the $N = 2$ algebra commutes with J^3_0, from a given $SL(2,\mathbb{R})$ representation with spin j, we will obtain a different $N = 2$ representation for each value of m. The use of j, m labels is most convenient, because performing integer spectral flow in $N = 2$ just amounts to an integer shift in m, as we will see below.

For a given $SL(2,\mathbb{R})$ representation, we are interested in the characters

$$ch^{NS}_{j,m}(q,y) = q^{-\frac{c}{24}} \mathrm{Tr}_{NS}\; q^{L_0} y^{J^R_0} \tag{A.7}$$

$$ch^{\widetilde{NS}}_{j,m}(q,y) = q^{-\frac{c}{24}} \mathrm{Tr}_{NS}\; (-1)^F q^{L_0} y^{J^R_0} \tag{A.8}$$

$$ch^{R}_{j,m}(q,y) = q^{-\frac{c}{24}} \mathrm{Tr}_{R}\; q^{L_0} y^{J^R_0} \tag{A.9}$$

where the trace is taken on the Hilbert space of the coset. We will compute in all the cases the NS character first. The Ramond characters will be obtained by half-spectral flow [71] as

$$ch^R(\tau,\nu) = q^{\frac{c}{6}(\frac{1}{2})^2} y^{\frac{c}{3}\frac{1}{2}} ch^{NS}(\tau, \nu + \frac{\tau}{2}) \tag{A.10}$$

where $q = e^{i2\pi\tau}, y = e^{i2\pi\nu}$. As for the $\widetilde{\mathrm{NS}}$ characters, it is easy to see that they are given by

$$ch^{\widetilde{NS}}_{j,m}(\tau,\nu) = e^{i\pi Q_{j,m}} ch^{NS}_{j,m}(\tau, \nu - 1/2) \tag{A.11}$$

where $Q_{j,m}$ is the $U(1)$ charge of the highest weight state. The NS characters can be obtained from

$$\chi_j(q,x,y) = \mathrm{Tr}\; q^{L_0 - \frac{c+3/2}{24}}\; x^{J^3_0} y^{J^R_0} = \sum_m x^m\, \xi_m(q)\, ch^{NS}_{j,m}(q,y) \tag{A.12}$$

where

$$\xi_m(q) = q^{-\frac{3}{48} - \frac{m^2}{k}} \prod_{n=1}^{\infty} \frac{(1 + q^{n-\frac{1}{2}})}{(1 - q^n)} \tag{A.13}$$

The trace in (A.12) is taken on the whole NS Hilbert space of the supersymmetric $SL(2,\mathbb{R})_k$ model and J_0^R is well-defined even before going to the coset. We have expanded $\chi_j(q,x,y)$ into terms with definite J_0^3 charge m and factored in each such term a supersymmetric $U(1)$ character $\xi_m(q)$ with highest weight $\Delta = -\frac{m^2}{k}$, generated by the modes of J^3, ψ^3.

The computation of $\chi_j(q,x,y)$ itself goes as follows. In the factorized $SL(2,\mathbb{R})_{k+2} \otimes \{\psi^a\}$ theory we can easily compute

$$\chi_j(q,z,w) = q^{-\frac{c+3/2}{24}} \ \mathrm{Tr}\ q^{L_0} z^{j_0^3} \times \mathrm{Tr}\ q^{L_0} w^{\hat{j}_0^3} \,, \tag{A.14}$$

and then from (A.2) and (A.3) it follows that $\chi_j(q,x,y)$ is given from $\chi_j(q,z,w)$ by the replacements

$$\begin{aligned} z \ &\to xy^{\frac{2}{k}} \,, \\ w &\to xy^{\frac{k+2}{k}} \,. \end{aligned} \tag{A.15}$$

In (A.14), the trace over the fermions NS Hilbert space is

$$\begin{aligned} \mathrm{Tr}\ q^{L_0} w^{\hat{j}_0^3} &= \prod_{n=1}^{\infty} (1+q^{n-\frac{1}{2}})(1+wq^{n-\frac{1}{2}})(1+w^{-1}q^{n-\frac{1}{2}}) \\ &= \prod_{n=1}^{\infty} \frac{(1+q^{n-\frac{1}{2}})}{(1-q^n)} \sum_{p\in\mathbb{Z}} w^p q^{\frac{p^2}{2}} \,. \end{aligned} \tag{A.16}$$

For the first factor in (A.14), we should consider the different possible representations of the j^a algebra. We take $q = e^{i2\pi\tau}, y = e^{i2\pi\nu}$.

Continuous representations

The j^a representations are built by acting with $j^a_{n<0}$ on representations of the zero modes j^a_0. For the continuous representations we have $j = \frac{1}{2} + iP$, $P \in \mathbb{R}^+$. The $SL(2,\mathbb{R})$ primaries are $|P,m\rangle$, with $m = r+\alpha$, $r \in \mathbb{Z}$, $\alpha \in [0,1)$ and conformal weight

$$-\frac{j(j-1)}{k} = \frac{\frac{1}{4}+P^2}{k} \,. \tag{A.17}$$

These primaries of j^a are multiplied by the fermionic NS vacuum to get the full primaries, so $j_0^3 = J_0^3 = m$ on these states. Considered as primaries of $N=2$, they give rise to unitary $N=2$ representations [23] with (see (A.3))

$$h_{P,m} = \frac{\frac{1}{4}+P^2+m^2}{k} \,, \qquad Q_m = \frac{2m}{k} \tag{A.18}$$

The character of the susy $SL(2,\mathbb{R})$ representation is

$$\begin{aligned} \chi_j(q,z,w) &= q^{-\frac{c+3/2}{24}} \ \mathrm{Tr}\ q^{L_0} z^{j_0^3} \times \mathrm{Tr}\ q^{L_0} w^{\hat{j}_0^3} \,, \\ &= q^{-\frac{c+3/2}{24}} q^{\frac{1/4+P^2}{k}} \sum_{r,p\in\mathbb{Z}} z^{\alpha+r} w^p q^{\frac{p^2}{2}} \prod_{n=1}^{\infty} \frac{(1+q^{n-\frac{1}{2}})}{(1-q^n)^4} \end{aligned} \tag{A.19}$$

and applying the steps described above it is immediate to obtain the $N=2$ characters

$$\begin{aligned} ch_c^{NS}(P,m;\tau,\nu) &= \text{Tr}\ q^{L_0-c/24}\, y^{J_0^R} \\ &= q^{\frac{P^2+m^2}{k}} y^{\frac{2m}{k}} \frac{\vartheta_3(\tau,\nu)}{\eta(\tau)^3} \end{aligned} \tag{A.20}$$

$$ch_c^{\widetilde{NS}}(P,m;\tau,\nu) = q^{\frac{P^2+m^2}{k}} y^{\frac{2m}{k}} \frac{\vartheta_4(\tau,\nu)}{\eta(\tau)^3} \tag{A.21}$$

$$ch_c^{R}(P,m';\tau,\nu) = q^{\frac{P^2+m'^2}{k}} y^{\frac{2m'}{k}} \frac{\vartheta_2(\tau,\nu)}{\eta(\tau)^3} \tag{A.22}$$

where $m' = m+\frac{1}{2}$ when the Ramond character is obtained by half-spectral flow from (A.20).

Discrete representations

Let us consider discrete lowest-weight representations $\mathcal{D}_j^+$ of j^a, with primaries $|j,r\rangle$, with $m = j+r$ and $r \in \mathbb{Z}, r > 0$, and with $j \in \mathbb{R}, j > 0$. For $0 < j < \frac{k+2}{2}$ they give rise to $N=2$ unitary representations [72, 23, 73]. This bound is further constrained in physical settings to be $\frac{1}{2} < j < \frac{k+1}{2}$. For each value of $m = j+r, r \in \mathbb{Z}$ the $N=2$ primaries are

$$\begin{array}{ll} r \geq 0 & |j, j+r\rangle \\ r < 0 & (j^-_{-1})^{-r-1} \psi^-_{-\frac{1}{2}} |j,j\rangle \end{array} \tag{A.23}$$

The quantum numbers are:

$$\begin{array}{llll} r \geq 0 & h_{j,r} = \frac{-j(j-1)+(j+r)^2}{k} & & Q_{j,r} = \frac{2(j+r)}{k} \\ r < 0 & h_{j,r} = \frac{-j(j-1)+(j+r)^2}{k} - r - \frac{1}{2} & & Q_{j,r} = \frac{2(j+r)}{k} - 1 \\ & \phantom{h_{j,r}} = \frac{-(\frac{k+2}{2}-j)(\frac{k+2}{2}-j-1)+(\frac{k+2}{2}-j-r-1)^2}{k} & & \phantom{Q_{j,r}} = -\frac{2(\frac{k+2}{2}-j-r-1)}{k} \end{array} \tag{A.24}$$

The second line in the $r < 0$ case shows that these states are similar to the $r \geq 0$ states when built from $\mathcal{D}^-_{\frac{k+2}{2}-j}$. Note that $r = 0, -1$ correspond to chiral and anti-chiral primaries respectively, as follows from $h_{j,0} = Q_{j,0}/2$ and $h_{j,-1} = -Q_{j,-1}/2$. These states are mapped to fermionic null states of relative charge ± 1 along the spectral flow orbit.

To compute the characters we start with

$$\begin{aligned} \chi_j(q,z,w) &= q^{-\frac{c}{24}}\ \text{Tr}\ q^{L_0} z^{j_0^3} \times \text{Tr}\ q^{L_0} w^{\hat{J}_0^3} \\ &= \frac{q^{-\frac{c}{24}-\frac{j(j-1)}{k}} z^j}{\prod_{n=1}^\infty (1-zq^{n-1})(1-z^{-1}q^n)(1-q^n)} \times \text{Tr}\ q^{L_0} w^{\hat{J}_0^3} \end{aligned} \tag{A.25}$$

Using[20]

$$\frac{1}{\prod_{n=1}^\infty (1-q^{n-1}z)(1-q^n z^{-1})} = \frac{1}{\prod_{n=1}^\infty (1-q^n)^2} \sum_{t=-\infty}^{\infty} z^t S_t(q)\,, \tag{A.26}$$

where

$$S_t(q) = \sum_{s=0}^\infty (-1)^s q^{\frac{1}{2}s(s+2t+1)} \tag{A.27}$$

[20] See [74] for a proof of this identity.

the character (A.25) can be expanded into

$$\chi_j(q,z,w) = q^{-\frac{c}{24}-\frac{j(j-1)}{k}} \prod_{n=1}^{\infty}(1+q^{n-\frac{1}{2}}) \sum_{p,t\in\mathbb{Z}} \frac{S_t(q)\, q^{\frac{p^2}{2}}\, z^{j+t}\, w^p}{\prod_{n=1}^{\infty}(1-q^n)^4} \tag{A.28}$$

After the replacement (A.15), and defining $r = t+p$, we get the decomposition as in (A.12)

$$\chi_j(q,x,y) = \sum_{r\in\mathbb{Z}} x^{j+r}\, \xi_{j+r}(q) \times \frac{q^{\frac{-(j-1/2)^2+(j+r)^2}{k}} y^{\frac{2(j+r)}{k}}}{\eta(\tau)^3} \sum_{p\in\mathbb{Z}} S_{r-p}(q) y^p q^{\frac{p^2}{2}} \,. \tag{A.29}$$

Using

$$\sum_{p\in\mathbb{Z}} S_{r-p}(q)\, y^p q^{\frac{p^2}{2}} = \frac{\sum_{p\in\mathbb{Z}} y^p q^{\frac{p^2}{2}}}{1+yq^{r+1/2}} = \frac{\vartheta_3(\tau,\nu)}{1+yq^{r+1/2}} \tag{A.30}$$

we get finally

$$\begin{aligned} ch_d^{NS}(j,r;\tau,\nu) &= \mathrm{Tr}\; q^{L_0-c/24} y^{J_0^R} \\ &= q^{-\frac{(j-1/2)^2}{k}+\frac{(j+r)^2}{k}} y^{\frac{2(j+r)}{k}} \frac{1}{1+yq^{1/2+r}} \frac{\vartheta_3(\tau,\nu)}{\eta(\tau)^3}\,, \\ &= q^{-\frac{(j-1/2)^2}{k}+\frac{(j+r)^2}{k}-r-\frac{1}{2}} y^{\frac{2(j+r)}{k}-1} \frac{1}{1+y^{-1}q^{-1/2-r}} \frac{\vartheta_3(\tau,\nu)}{\eta(\tau)^3}\,. \end{aligned} \tag{A.31}$$

We wrote the characters in two forms, each form reflecting the structure of the representation for a different range of r (see (A.23)). For the other sectors we get

$$ch_d^{\widetilde{NS}}(j,r;\tau,\nu) = q^{-\frac{(j-1/2)^2}{k}+\frac{(j+r)^2}{k}} y^{\frac{2(j+r)}{k}} \frac{1}{1-yq^{1/2+r}} \frac{\vartheta_4(\tau,\nu)}{\eta(\tau)^3} \tag{A.32}$$

$$ch_d^{R}(j,r';\tau,\nu) = q^{-\frac{(j-1/2)^2}{k}+\frac{(j+r')^2}{k}} y^{\frac{2(j+r')}{k}} \frac{1}{1+yq^{1/2+r'}} \frac{\vartheta_2(\tau,\nu)}{\eta(\tau)^3} \tag{A.33}$$

where $r' = r+\frac{1}{2}$ when the Ramond character is obtained by half-spectral flow from (A.31).

Finite dimensional representations

In this case the spin takes the values $j = -\frac{(u-1)}{2}, u = 1,2....$ The highest weights are u-dimensional representations of j_0^a given by $|j,m\rangle$, with $m = j, j+1, \cdots -j$. Only for $u=1$ the induced $N=2$ representation is unitary. For every $m = j+r, r\in\mathbb{Z}$ the $N=2$ primaries are

$$\begin{array}{ll} r<0 & (j_{-1}^{-})^{-r-1}\psi^{-}_{-\frac{1}{2}}|j,j\rangle \\ 0\le r\le u-1 & |j,j+r\rangle \\ r>u-1 & (j_{-1}^{+})^{r-u}\psi^{+}_{-\frac{1}{2}}|j,-j\rangle \end{array} \tag{A.34}$$

with quantum numbers

$$\begin{array}{lll} r<0 & h_{j,r} = \frac{-j(j-1)+(r+j)^2}{k} - r - \frac{1}{2} & Q_{j,r} = \frac{2(r+j)}{k} - 1 \\ 0\le r\le u-1 & h_{j,r} = \frac{-j(j-1)+(r+j)^2}{k} & Q_{j,r} = \frac{2(r+j)}{k} \\ r>u-1 & h_{j,r} = \frac{-j(j-1)+(r+j)^2}{k} + r - u + \frac{1}{2} & Q_{j,r} = \frac{2(r+j)}{k} + 1 \end{array} \tag{A.35}$$

In order to compute the characters, we start with

$$\begin{aligned}\chi_j(q,z,w) &= q^{-\frac{c}{24}}\ \mathrm{Tr}\ q^{L_0} z^{j_0^3} \times \mathrm{Tr}\ q^{L_0} w^{\hat{J}_0^3} \\ &= \frac{q^{-\frac{c}{24}-\frac{j(j-1)}{k}}\left(z^{-\frac{(u-1)}{2}} - z^{\frac{u+1}{2}}\right)}{\prod_{n=1}^{\infty}(1-zq^{n-1})(1-z^{-1}q^n)(1-q^n)} \times \prod_{n=1}^{\infty} \frac{(1+q^{n-\frac{1}{2}})}{(1-q^n)} \sum_{p\in\mathbb{Z}} w^p q^{\frac{p^2}{2}}\end{aligned} \tag{A.36}$$

The expansion goes along similar lines to the discrete representations and is left as an exercise. The resulting $N=2$ characters are

$$\begin{aligned}ch_f^{NS}(u,r;\tau,\nu) &= \mathrm{Tr}\ q^{L_0-c/24} z^{J_0^R} \\ &= q^{-\frac{(j-1/2)^2}{k}+\frac{(r+j)^2}{k}-r-1/2} y^{\frac{2(r+j)}{k}-1} \quad \frac{(1-q^u)}{(1+y^{-1}q^{-1/2-r})(1+y^{-1}q^{u-1/2-r})} \frac{\vartheta_3(\tau,\nu)}{\eta(\tau)^3}\,, \\ &= q^{-\frac{(j-1/2)^2}{k}+\frac{(r+j)^2}{k}} y^{\frac{2(r+j)}{k}} \quad \frac{(1-q^u)}{(1+yq^{+1/2+r})(1+y^{-1}q^{u-1/2-r})} \frac{\vartheta_3(\tau,\nu)}{\eta(\tau)^3}\,, \\ &= q^{-\frac{(j-1/2)^2}{k}+\frac{(r+j)^2}{k}+r+2j-\frac{1}{2}} y^{\frac{2(r+j)}{k}+1} \quad \frac{(1-q^u)}{(1+yq^{1/2+r})(1+yq^{r-u+1/2})} \frac{\vartheta_3(\tau,\nu)}{\eta(\tau)^3}\,.\end{aligned} \tag{A.37}$$

Again, we expressed the characters in forms reflecting the structure of the representation for different ranges of r. These characters can also be expressed as

$$\begin{aligned}ch_f^{NS}(u,r;\tau,\nu) &= ch_d^{NS}(j,r;\tau,\nu) - ch_d^{NS}(-j+1,r-u;\tau,\nu) \\ &= \frac{\vartheta_3(\tau,\nu)}{\eta(\tau)^3} q^{\frac{s^2-su}{k}} y^{\frac{2s-u}{k}} \left[\frac{1}{1+yq^s} - \frac{1}{1+yq^{s-u}}\right]\end{aligned} \tag{A.38}$$

with $s = r+\frac{1}{2}$. For the other sectors we get

$$\begin{aligned}ch_f^{\widetilde{NS}}(u,r;\tau,\nu) &= q^{-\frac{(j-1/2)^2}{k}+\frac{(r+j)^2}{k}} y^{\frac{2(r+j)}{k}} \quad \frac{(1-q^u)}{(1-yq^{+1/2+r})(1-y^{-1}q^{u-1/2-r})} \frac{\vartheta_4(\tau,\nu)}{\eta(\tau)^3} \\ ch_f^{R}(u,r';\tau,\nu) &= q^{-\frac{(j-1/2)^2}{k}+\frac{(r'+j)^2}{k}} y^{\frac{2(r'+j)}{k}} \quad \frac{(1-q^u)}{(1+yq^{+1/2+r'})(1+y^{-1}q^{u-1/2-r'})} \frac{\vartheta_2(\tau,\nu)}{\eta(\tau)^3}\end{aligned} \tag{A.39}$$

where $r' = r+\frac{1}{2}$ when the Ramond character is obtained by half-spectral flow from (A.37).

Spectral flow

The $N=2$ algebra has the spectral flow automorphism [71]

$$\begin{aligned}L_n &\to L_n + wJ_n + \frac{c}{6}w^2\delta_{n,0} \\ J_n &\to J_n + \frac{c}{3}\delta_{n,o} \\ G_m^{\pm} &\to G_{m\pm w}^{\pm}\end{aligned} \tag{A.40}$$

which maps a representation into another one. The spectrum of the flowed representations is obtained by measuring the flowed L_0, J_0 on the original representation. The character of the spectrally flowed representation is then given by

$$ch(\tau,\nu) \to q^{\frac{c}{6}w^2}\ y^{\frac{c}{3}w}\ ch(\tau,\nu+w\tau) \tag{A.41}$$

Using (E.5), it is immediate to verify that in the characters considered above the spectral flow by w integer units is equivalent to a shift $m \to m + w$, thus verifying our claim that m keeps track of the $N = 2$ spectral flow orbit.

Finally, notice that in the discrete and finite cases, $SL(2,\mathbb{R})$ representations in which the generators act freely give rise to an $N = 2$ representations with null descendents due to the semi-infinite/finite base in $SL(2,\mathbb{R})$. To get some insight into the form of the characters, note that the free action of the modes $L_{-n}, J_{-n}, G^{\pm}_{-n+1/2}$ on the highest weight state gives the contribution (in the NS sector),

$$\prod_{n=1}^{\infty} \frac{(1+q^{n-\frac{1}{2}}y)(1+q^{n-\frac{1}{2}}y^{-1})}{(1-q^n)^2} = q^{1/8}\frac{\vartheta_3(\tau,\nu)}{\eta(\tau)^3} \tag{A.42}$$

In the discrete and finite representations the character is further modded out by the null descendents.

B. Changing basis in $SL(2,\mathbb{R})$

In this appendix we recall how to Fourier transform the one-point functions of [49] into the form in which they were used in [43], and in the bulk of our paper, in our conventions.

Reflection amplitude

We use a quadratic Casimir for $SL(2,R)$ representation of the form:

$$c_2 = -j(j-1) \tag{B.1}$$

and we adopt conventions in which the bulk two-point functions are:

$$\begin{aligned} \langle \Phi^j(x^1,\bar{x}^1)\Phi^{j'}(x^2,\bar{x}^2)\rangle &= |z_{12}|^{-4\Delta_j}\left(\delta^2(x^1-x^2)\delta(j+j'-1) + \frac{B(j)}{|x_{12}|^{4j}}\delta(j-j')\right) \\ B(j) &= \frac{k}{\pi}\nu^{1-2j}\frac{\Gamma(1-\frac{2j-1}{k})}{\Gamma(\frac{2j-1}{k})} \end{aligned} \tag{B.2}$$

which leads after Fourier transformation

$$\Phi^j_{nw} = \frac{1}{4\pi^2}\int d^2x\, x^{j-1+m}\bar{x}^{j-1+\bar{m}}\Phi^j(x,\bar{x}) \tag{B.3}$$

to

$$\begin{aligned} \langle \Phi^j_{nw}\Phi^{j'}_{n'w'}\rangle &= |z_{12}|^{-4\Delta_j}\delta_{n+n'}\delta_{w+w'}\left(\delta(j+j'-1) + R(j,n,w)\delta(j-j')\right) \\ R(j,n,w) &= \nu^{1-2j}\frac{\Gamma(-2j+1)\Gamma(j+m)\Gamma(j-\bar{m})\Gamma(1+\frac{1-2j}{k})}{\Gamma(2j-1)\Gamma(-j+1+m)\Gamma(-j+1-\bar{m})\Gamma(1-\frac{1-2j}{k})}. \end{aligned} \tag{B.4}$$

In the super-coset, we identify the elliptic eigenvalues of $SL(2,R)$ with the geometric angular momentum and winding by:

$$\begin{aligned} m &= (n+kw)/2 \\ \bar{m} &= (-n+kw)/2. \end{aligned} \tag{B.5}$$

Localized branes

Starting from the one-point function for spherical branes in H_3 [49]:

$$\langle \Phi^j(x|z)\rangle_s = -\frac{1}{2\pi}(1+x\bar{x})^{-2j}\frac{\Gamma(1+(-2j+1)/k)}{\Gamma(1-1/k)}\frac{\sin s(-2j+1)}{\sin s}\nu^{-j+1}|z-z'|^{-2\Delta_j}, \tag{B.6}$$

we find its Fourier transform:

$$\begin{aligned}\langle \Phi^j_{np}(z)\rangle_s &= \int d^2x e^{inarg(x)}|x|^{2j-2-ip}\langle \Phi^j(x|z)\rangle_s \\ &= \frac{1}{2}\delta_{n,0}\frac{\Gamma(j-ip/2)\Gamma(j+ip/2)\Gamma(1+1/k)}{\Gamma(2j-1)\Gamma(1-(-2j+1)/k)}\frac{\sin\pi/k \sin s(-2j+1)}{\sin s\sin\pi(-2j+1)/k},\end{aligned} \tag{B.7}$$

where we made use of the integral:

$$\int_0^\infty drr^{2j-1-ip}(1+r^2)^{-2j} = \frac{1}{2}\frac{\Gamma(j-ip/2)\Gamma(j+ip/2)}{\Gamma(2j)}. \tag{B.8}$$

Up to normalization, this corresponds to the one-point function in the bosonic and supersymmetric coset.

Extended AdS_2 branes

Starting from the one-point function for extended branes in H_3 (where $\sigma = \mathrm{sgn}(x+\bar{x})$):

$$\langle \Phi^j(x|z)\rangle_r = \frac{\mathcal{N}k}{\pi}|x-\bar{x}|^{-2j}\nu^{-j+1/2}\Gamma(1+(-2j+1)/k)e^{(1-2j)r\sigma}|z-z'|^{-2\Delta_j}, \tag{B.9}$$

we find the transformed coset one-point function:

$$\begin{aligned}\langle \Phi^j_{np}(z)\rangle_r &= \int dx^2 e^{inarg(x)}|x|^{2j-2-ip}\langle \Phi^j(x|z)\rangle_r \\ &= 2\pi k\delta(p)\mathcal{N}\nu^{-j+1/2}\frac{\Gamma(-2j+1)\Gamma(1+(-2j+1)/k)}{\Gamma(-j+1+n/2)\Gamma(-j+1-n/2)}((-1)^n e^{r(-2j+1)}+e^{-r(-2j+1)}).\end{aligned} \tag{B.10}$$

We used the integrals:

$$\begin{aligned}\int_{-\pi/2}^{\pi/2} d\phi e^{in\phi}(2\cos\phi)^{-2j} &= \frac{\pi\,\Gamma(-2j+1)}{\Gamma(1-j+n/2)\Gamma(1-j-n/2)} \\ \int_0^\infty drr^{-1-ip} &= 2\pi\delta(p).\end{aligned} \tag{B.11}$$

Thus we have reviewed the connection between the one-point functions in [49] and the one-point functions for the coset [43] in our conventions.

C. Cardy condition for D0 branes in R/$\widetilde{\mathrm{NS}}$ sectors

In this appendix we will provide some more details of the Cardy computation for D0 branes in the R and $\widetilde{NS}$ sectors, along the lines of section 4.

Let us start Ramond sector in the open string channel. For the partition function in the $(u,1)$ case we take

$$\begin{aligned} Z^R_{u,1}(\tau,\nu) &= \sum_{r\in\mathbb{Z}+\frac{1}{2}} ch^R_f(u,r;\tau,\nu) \\ &= \frac{\vartheta_2(\tau,\nu)}{\eta(\tau)^3}\sum_{s\in\mathbb{Z}}\frac{1}{1+yq^s}\left(q^{\frac{s^2-su}{k}}y^{\frac{2s-u}{k}} - q^{\frac{s^2+su}{k}}y^{\frac{2s+u}{k}}\right), \end{aligned} \tag{C.1}$$

where we have summed over the whole spectral flow orbit of the $N=2$ Ramond characters associated with the u-dimensional representation of $SL(2,\mathbb{R})$. For the general (u,u') case we make a sum as in (4.6). The computation is very similar to the NS/NS case, so we will only indicate the major steps.

We start with the modular transform of $Z^R_{u,1}(\tau,\nu)$, given by

$$\begin{aligned} e^{-i\pi\frac{c}{3}\frac{\nu^2}{\tau}}\quad Z^R_{u,1}(-\frac{1}{\tau},\frac{\nu}{\tau}) &= \frac{\vartheta_4(\tau,\nu)}{\eta(\tau)^3}\quad\times \\ \frac{1}{2i\pi}\left[\int_{\mathcal{C}_{-\epsilon}}+\int_{\mathcal{C}_{+\epsilon}}\right] dZ\ (-i\pi)\, e^{\pi Z+\frac{2i\pi\tau}{k}Z^2}\ & \frac{\sinh(2\pi\frac{Z}{k}u)}{\cosh(\pi Z)}\frac{e^{i\pi(i\tau Z-\nu+\frac{1}{2})}}{\cos\pi(i\tau Z-\nu+\frac{1}{2})} \end{aligned} \tag{C.2}$$

The contour of integration is the same as in section 4, and the integrand has poles at $\nu - i\tau Z = s\in\mathbb{Z}$. Expanding the last factor of the integrand as a sum over w as in (4.9-4.10), taking $\varepsilon\to 0$ and shifting the contour by $\frac{iwk}{2}-i\nu_2$ in each term, we arrive, for general (u,u'), to a decomposition

$$e^{-i\pi\frac{c}{3}\frac{\nu^2}{\tau}}Z^R_{u,u'}(-\frac{1}{\tau},\frac{\nu}{\tau}) = Z^{R,c}_{u,u'}+Z^{R,d}_{u,u'} \tag{C.3}$$

where the first term is an integral over the shifted contour and the second is the sum of poles picked during the shift. These terms are

$$\begin{aligned} Z^{R,c}_{u,u'} &= \sum_{w\in\mathbb{Z}}\int_0^{+\infty}dP\frac{2\sinh(2\pi P)\sinh(2\pi\frac{P}{k}u)\sinh(2\pi\frac{P}{k}u')}{(-1)^{w(u+u'+1)}[\cosh(2\pi P)+\cos(\pi kw)]\sinh(2\pi\frac{P}{k})} \\ &\qquad\qquad\times\ q^{\frac{P^2+(kw/2)^2}{k}}y^{\frac{2(kw/2)}{k}}\frac{\vartheta_4(\tau,\nu)}{\eta(\tau)^3} \\ &= \sum_{w\in\mathbb{Z}}\int_0^{+\infty}dP\ \Psi^{\widetilde{NS}}_u\left(\frac{1}{2}-iP,w\right)\Psi^{\widetilde{NS}}_{u'}\left(\frac{1}{2}+iP,w\right)\ ch^{\widetilde{NS}}_c(P,\frac{wk}{2};\tau,\nu) \end{aligned} \tag{C.4}$$

and

$$\begin{aligned} Z^{R,d}_{u,u'} &= \sum_{r\in\mathbb{Z}}\frac{2\sin\left(\frac{2\pi}{k}(2j_r-1)u\right)\sin\left(\frac{2\pi}{k}(2j_r-1)u'\right)}{(-1)^{w_r(u+u'+1)}\sin\left(\frac{2\pi}{k}(2j_r-1)\right)}\ \frac{\vartheta_4(\tau,\nu)}{\eta(\tau)^3}\frac{y^{w_r}q^{sw_r-\frac{s^2}{k}}}{1-yq^s} \\ &= 2\pi\sum_{r\in\mathbb{Z}}Res\left[\Psi^{\widetilde{NS}}_u(-j_r+1,w_r)\Psi^{\widetilde{NS}}_{u'}(j_r,w_r)\right]\ ch^{\widetilde{NS}}_d(j_r,r;\tau,\nu) \end{aligned} \tag{C.5}$$

where j_r and w_r are defined in the same way as in section 4 and $s = r + \frac{1}{2}$ as usual. We see thus that the Cardy condition is verified.

For the open string partition function in the $\widetilde{NS}$ sector, which computes the open string Witten index [75], we start with

$$\begin{aligned} Z_{u,1}^{\widetilde{NS}}(\tau,\nu) &= \sum_{r\in\mathbb{Z}} ch_f^{\widetilde{NS}}(u,r;\tau,\nu) \\ &= \frac{\vartheta_4(\tau,\nu)}{\eta(\tau)^3} \sum_{s\in\mathbb{Z}+\frac{1}{2}} \frac{1}{1-yq^s}\left(q^{\frac{s^2-su}{k}} y^{\frac{2s-u}{k}} - q^{\frac{s^2+su}{k}} y^{\frac{2s+u}{k}}\right) \end{aligned} \tag{C.6}$$

Its modular transform is

$$\begin{aligned} e^{-i\pi\frac{c}{3}\frac{\nu^2}{\tau}} \quad Z_{u,1}^{\widetilde{NS}}(-\frac{1}{\tau},\frac{\nu}{\tau}) &= \frac{\vartheta_2(\tau,\nu)}{\eta(\tau)^3} \\ \times \quad \frac{1}{2i\pi}\left[\int_{\mathcal{C}_{-\epsilon}} + \int_{\mathcal{C}_{+\epsilon}}\right] dZ\ (-i\pi)\, e^{\pi Z + \frac{2i\pi\tau}{k}Z^2}\ &\frac{\sinh(2\pi\frac{Z}{k}u)}{\sinh(\pi Z)} \frac{e^{i\pi(i\tau Z-\nu)}}{\cos\pi(i\tau Z-\nu)} \end{aligned} \tag{C.7}$$

with the same contour as before, but now the integrand has poles at $\nu - i\tau Z = s \in \mathbb{Z} + \frac{1}{2}$. Expanding again the last factor of the integrand as in (4.9-4.10), taking $\varepsilon \to 0$ and shifting the contour by $\frac{iwk}{2} - i\nu_2$ in each term, we arrive, for general (u, u'), to

$$e^{-i\pi\frac{c}{3}\frac{\nu^2}{\tau}} Z_{u,u'}^{\widetilde{NS}}(-\frac{1}{\tau},\frac{\nu}{\tau}) = Z_{u,u'}^{\widetilde{NS},c} + Z_{u,u'}^{\widetilde{NS},d} \tag{C.8}$$

where

$$\begin{aligned} Z_{u,u'}^{\widetilde{NS},c} &= \sum_{w\in\mathbb{Z}} \int_0^{+\infty} dP \frac{2\sinh(2\pi P)\sinh(2\pi\frac{P}{k}u)\sinh(2\pi\frac{P}{k}u')}{(-1)^{w(u+u')}[\cosh(2\pi P)-\cos(\pi kw)]\sinh(2\pi\frac{P}{k})} \\ &\qquad\qquad \times\ q^{\frac{P^2+(kw/2)^2}{k}} y^{\frac{2(kw/2)}{k}} \frac{\vartheta_2(\tau,\nu)}{\eta(\tau)^3} \\ &= \sum_{w\in\mathbb{Z}} \int_0^{+\infty} dP\ \Psi_u^{R^-}\left(\frac{1}{2}-iP,-w\right) \Psi_{u'}^{R^+}\left(\frac{1}{2}+iP,w\right)\ ch_c^R(P,\frac{wk}{2};\tau,\nu) \end{aligned} \tag{C.9}$$

and

$$\begin{aligned} Z_{u,u'}^{\widetilde{NS},d} &= \sum_{r\in\mathbb{Z}+\frac{1}{2}} (-1)^{w_r(u+u')} \frac{2\sin\left(\frac{\pi}{k}(2j_r-1)u\right)\sin\left(\frac{\pi}{k}(2j_r-1)u'\right)}{\sin\left(\frac{\pi}{k}(2j_r-1)\right)} \frac{y^{w_s}q^{sw_s-\frac{s^2}{k}}}{1+yq^s} \frac{\vartheta_2(\tau,\nu)}{\eta(\tau)^3} \\ &= 2\pi \sum_{r\in\mathbb{Z}+\frac{1}{2}} Res\left[\Psi_u^{R^-}(-j_r+1,-w_r)\Psi_{u'}^{R^+}(j_r,w_r)\right]\ ch_d^R(j_r,r;\tau,\nu)\,, \end{aligned} \tag{C.10}$$

with j_r and w_r defined as in section 4, but now taking $r \in \mathbb{Z} + \frac{1}{2}$.

In both $Z_{u,u'}^{R,d}$ and $Z_{u,u'}^{\widetilde{NS},d}$ the residues are computed when considering the bracketed expressions as analytical functions of j.

D. Embedding $N = 2$ into $SL(2, \mathbb{R})$

Our goal in this appendix is to elaborate on the relation between the $N = 2, c > 3$ chiral algebra and the $SL(2, \mathbb{R})$ algebra, showing how the former always lead to the latter. These ideas go back to [23].

Consider an $N = 2$ algebra with $c > 3$. We can write the supercurrents and the $U(1)$ R-current in the following form:

$$\begin{aligned} G^{\pm} &= \sqrt{\frac{2c}{3}}\pi^{\pm} e^{\pm i\sqrt{\frac{3}{c}}\phi} \\ J^R &= i\sqrt{\frac{c}{3}}\partial\phi \end{aligned} \tag{D.1}$$

where we made use of a canonically normalized scalar ϕ and the fact that the supercurrents carry $U(1)$ R-charge ± 1. The fields $\pi^{\pm}$ are the first parafermionic currents of $SL(2, \mathbb{R})/U(1)$. We now observe that if we add a trivial auxiliary $U(1)$ to the theory, parameterized by an anti-hermitean scalar field T, we can define the following currents:

$$\begin{aligned} j^3 &= i\sqrt{\frac{k+2}{k}}\partial\phi + \frac{(k+2)}{\sqrt{2k}}\partial T = J^R + \frac{(k+2)}{\sqrt{2k}}\partial T \\ j^{\pm} &= \sqrt{k+2}\,\pi^{\pm} e^{\mp\sqrt{\frac{2}{k+2}}\left(i\sqrt{\frac{2}{k}}\phi + \sqrt{\frac{k+2}{k}}T\right)} \end{aligned} \tag{D.2}$$

which satisfy an $SL(2, R)$ algebra at level $k + 2$:[21]

$$\begin{aligned} j^3(z)j^3(w) &\sim -\frac{k+2}{2(z-w)^2}, \\ j^3(z)j^{\pm}(w) &\sim \pm\frac{j^{\pm}(w)}{z-w}, \\ j^{+}(z)j^{-}(w) &\sim \frac{k+2}{(z-w)^2} - \frac{2j^3(w)}{z-w}. \end{aligned} \tag{D.4}$$

To make use of the purely bosonic current algebra when solving the theory, we will need to introduce primary fields for the $N = 2$ algebra combined with the $U(1)$ that transform as standard representations of the $SL(2, R)$ current algebra. This can be achieved as follows. When we denote $Z_{j;r,\bar{r}}$ a primary of the $N = 2$ algebra with conformal weight $\Delta_{j,r} = -j(j-1)/k + m^2/k$ and $U(1)$ R-charge $m = j + r$, we can introduce the new primaries $\hat{Z}_{j,r,\bar{r}}$:

$$\hat{Z}_{j,r,\bar{r}} = Z_{j,r,\bar{r}} e^{\alpha_{j,r}T_L + \alpha_{j,\bar{r}}T_R}, \tag{D.5}$$

[21] If we wish, we can add a super-partner for the boson T, and view the linear combination of $\partial\phi$ and ∂T orthogonal to I^3 as bosonized complex fermion:

$$i\sqrt{\frac{k+2}{k}}\partial\phi + \sqrt{\frac{2}{k}}\partial T =: \psi^+\psi^- : . \tag{D.3}$$

The three fermions then complete an $N = 1$ supersymmetric $SL(2, \mathbb{R})_k$ theory, containing a purely bosonic $SL(2, \mathbb{R})_{k+2}$.

where we choose $\alpha_{j,r}$ such that

$$\Delta(\hat{Z}_{j,r,\bar{r}}) = -\frac{j(j-1)}{k} = \Delta_{j,r} - \frac{\alpha_{j,r}^2}{2} \tag{D.6}$$

(i.e. the momentum of the boson T is coupled to the $N=2$ $U(1)$ R-charge). The new primaries can be checked to transform in a standard fashion as operators of $SL(2,R)$, in a basis labeled by the spectrum of an elliptic generator. To go to the hyperbolic basis we can Fourier transform:

$$\Phi_j(z,\bar{z},x,\bar{x}) = \sum_{m,\bar{m}} \hat{Z}_{j,r,\bar{r}}\ x^{j-1+m}\bar{x}^{j-1+\bar{m}}. \tag{D.7}$$

We can now compute correlators of these fields, which are linear combinations of primaries of the $N=2$ algebra "dressed" with an orthogonal and free $U(1)$, using a full chiral $SL(2,R)$. Thus, if the solution of the H_3^+ theory is purely based on symmetries, this holds for any $N=2$ theory with $c>3$. [22]

We note also that the construction above shows that the "x-variables" basis will remain an efficient formalism for computation as long as $N=2$ supersymmetry is preserved. This complements the remarks in [40] where it was suggested that these variables should be interpreted as parameterizing a new worldsheet, at any point in the moduli space.

E. Conventions

We take $q = e^{i2\pi\tau}, y = e^{i2\pi\nu}$, and we define:

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n) \tag{E.1}$$

$$\vartheta_2(\tau,\nu) = q^{\frac{1}{8}}y^{\frac{1}{2}}\prod_{n=1}^{\infty}(1-q^n)(1+q^n y)(1+q^{n-1}y^{-1}) = \sum_{n\in\mathbb{Z}+\frac{1}{2}} q^{\frac{n^2}{2}}y^n \tag{E.2}$$

$$\vartheta_3(\tau,\nu) = \prod_{n=1}^{\infty}(1-q^n)(1+q^{n-\frac{1}{2}}y)(1+q^{n-\frac{1}{2}}y^{-1}) = \sum_{n\in\mathbb{Z}} q^{\frac{n^2}{2}}y^n \tag{E.3}$$

$$\vartheta_4(\tau,\nu) = \prod_{n=1}^{\infty}(1-q^n)(1-q^{n-\frac{1}{2}}y)(1-q^{n-\frac{1}{2}}y^{-1}) = \sum_{n\in\mathbb{Z}}(-1)^n q^{\frac{n^2}{2}}y^n. \tag{E.4}$$

A shift in the second argument of the third theta-function can be compensated for as follows ($w\in\mathbb{Z}$):

$$\vartheta_3(\tau,\eta+w\tau) = q^{-\frac{w^2}{2}}y^{-w}\vartheta_3(\tau,\eta). \tag{E.5}$$

[22] We assume that the $N=2$ theory does not split into subtheories with $c<3$. This assumes the irreducibility of the representations space of the $N=2, c>3$ algebra. Otherwise, a similar $SU(2)$ algebra would be a more appropriate tool to solve the theory.

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